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INVESTIGATION OF A ONE-DIMENSIONAL MIXED PROBLEM FOR THIRD ORDER NONLINEAR DIFFERENTIAL EQUATIONS

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Abstract

This work is devoted to the study of generalized solution for one-dimensional mixed problem with non-self-adjoint boundary condition for a class of nonlinear differential equations. The solution of the problem under consideration is reduced to solving a system of certain nonlinear integro-differential equations.

Keywords: *nonlinear equation, mixed problem, generalized solution*

The work is devoted to the study of a generalized solution to the following one-dimensional mixed problem

$$\begin{cases} u_{tt}(t,x) - u_{txx}(t,x) = \\ = F(t,x,u(t,x),u_t(t,x),u_x(t,x),u_{tx}(t,x),u_{xx}(t,x)) \\ (0 \le t \le T, 0 \le x \le 1), & (1) \\ u(0,x) = \phi(x)\ (0 \le x \le 1),\ u_t(0,x) = \psi(x)\ (0 \le x \le 1), & (2) \\ u(t,0) = 0\ (0 \le t \le T),\ u_x(t,0) = u_x(t,1)\ (0 \le t \le T), & (3) \end{cases}$$

where $0 < T < +\infty; F, \phi, \psi$ are the given functions, $u(t,x)$ is a desired function, and under the generalized solution of problem (1)–(3) we understand the following:

Definition. Under the generalized solution of problem (1)–(3) we understand the function $u(t,x)$ having the following properties:

a) $u(t,x), u_x(t,x), u_t(t,x) \in C([0,T]\times[0,1])$;

$$u_{xx}(t,x), u_{tx}(t,x) \in C([0,T]; L_2(0,1));$$

б) all the conditions of (2) and (3) are satisfied in the usual sense;

в) the integral identity

$$\int_0^T\int_0^1 \left\{\left[u_t(t,x) - u_{xx}(t,x)\right]\cdot V_t(t,x) + F(u(t,x))\cdot V(t,x)\right\} dxdt + \\ + \int_0^1 [\psi(x) - \phi''(x)]\cdot V(0,x)dx = 0$$

is fulfilled for any function $V(t,x)$, having the properties

$$V(t,x) \in C([0,T]\times[0,1]), V_t(t,x) \in \\ \in L([0,T]; L_2(0,1)), V(T,x) = 0\ \forall x \in [0,1],$$

where

$$F(u(t,x)) \equiv \\ \equiv F(t,x,u(t,x),u_t(t,x),u_x(t,x),u_{tx}(t,x),u_{xx}(t,x)). \tag{4}$$

Few studies have been devoted to the investigation of one-dimensional non-self-adjoint mixed problems for third-order semilinear equations. However, self-adjoint mixed problems have been well studied (see Huntul M., Tekin I.,2022; Aliyev S., Aliyeva A., Abdullayeva G., 2019; Aliyev S., Aliyeva A.,2020; Aliyev S., Khudaverdiev K.,1990 and the references therein), in which the problem of existence and uniqueness in the corresponding spaces is investigated. In (Aliyev S., Heydarova M., Aliyeva A., 2024), studied the existence of a classical solution to a one-dimensional mixed problem for a certain semilinear equation, which is simpler than equation (1). But in (Aliyeva A., 2012) the generalized solution and in (Aliyev S., Aliyeva A., 2017) the almost everywhere solution of the considered mixed problem was investigated.

It is known (Ionkin N. I., 1977) that the systems

$$X_0(x)=x,\dots,X_{2k-1}(x)=x\cos 2\pi kx,\quad X_{2k}(x)=\sin 2\pi kx,\dots \tag{5}$$

and

$$Y_0(x)=2,\quad Y_{2k-1}(x)=4\cos 2\pi kx,\ Y_{2k}(x)=4(1-x)\sin 2\pi kx,\dots \tag{6}$$

form a biorthogonal system of functions in the space $L_2(0,1)$, and that system (7) forms a basis in the space $L_2(0,1)$. Then, it is obvious that each generalized solution of problem (1)-(3) has the form:

$$u(t,x)=\sum_{k=0}^{\infty}u_k(t)X_k(x), \tag{7}$$

where

$$u_k(t)=\int_0^1 u(t,x)Y_k(x)dx \quad (k=0,1,\dots;t\in[0,T]). \tag{8}$$

Now let's apply the formal scheme of the Fourier method to problem (1)-(3). Let $u(t,x)=\sum_{k=0}^{\infty}u_k(t)X_k(x)$ be any generalized solution to problem (1)-(3). Then it is obvious that for any fixed k $(k=0,1,\dots)$ and any $t\in[0,T]$:

$$\int_0^1 u_{tt}(t,x)Y_k(x)dx=\frac{d^2}{dt^2}\left(\int_0^1 u(t,x)Y_k(x)dx\right)=u_k''(t); \tag{9}$$

$$\int_0^1 u_{txx}(t,x)Y_k(x)dx=u_{tx}(t,x)Y_k(x)\Big|_{x=0}^{x=1}-\int_0^1 u_{tx}(t,x)Y_k'(x)dx=$$

$$=-\int_0^1 u_{tx}(t,x)Y_k'(x)dx=$$

$$=-u_t(t,x)Y_k'(x)\Big|_{x=0}^{x=1}+\int_0^1 u_t(t,x)Y_k''(x)dx=$$

$$=\int_0^1 u_t(t,x)Y_k''(x)dx=\frac{d}{dt}\left(\int_0^1 u(t,x)Y_k''(x)dx\right), \tag{10}$$

since

$$u_x(t,0)=u_x(t,1),u_{tx}(t,0)=u_{tx}(t,1),Y_k(0)=Y_k(1);$$
$$u(t,0)=0,u_t(t,0)=0,Y_k'(1)=0.$$

Further we will adopt the notation

$$F_k(u;t)\equiv\int_0^1 F(u(t,x))Y_k(x)dx \quad (k=0,1,\dots;t\in[0,T]), \tag{11}$$

where F is defined by (4).

From equation (1), multiplying it by the function $Y_k(x)$, integrating the resulting equality over x from 0 to 1, using relations (9),(10) and notations (11), we obtain that for any k $(k=0,1,\dots)$ and $t\in[0,T]$:

$$u_k''(t)-\frac{d}{dt}\left(\int_0^1 u(t,x)Y_k''(x)dx\right)=F_k(u;t).$$

Now, using relation (8) and notation (10), we obtain that for any $\forall t\in[0,T]$:

$$u_0''(t)=F_0(u;t), \tag{12}$$

$$u_{2k-1}''(t)+(2\pi k)^2\cdot u_{2k-1}'(t)=F_{2k-1}(u;t)\quad(k=1,2,\dots), \tag{13}$$

$$u_{2k}''(t)+(2\pi k)^2\cdot u_{2k}'(t)=$$
$$=-4\pi k\cdot u_{2k-1}'(t)+F_{2k}(u;t)\quad(k=1,2,\dots). \tag{14}$$

Thus, the general solution of equations (12) – (14) has the form:

$$u_0(t)=A_0+B_0\cdot t+\int_0^t(t-\tau)\cdot F_0(u;\tau)d\tau \quad (t\in[0,T]),$$

$$u_{2k-1}(t)=A_{2k-1}+B_{2k-1}\cdot e^{-(2\pi k)^2 t}+$$
$$+\frac{1}{(2\pi k)^2}\cdot\int_0^t\left[1-e^{-(2\pi k)^2(t-\tau)}\right]\cdot F_{2k-1}(u;\tau)d\tau,$$
$$(k=1,2,\dots;t\in[0,T])$$

$$u_{2k}(t)=A_{2k}+B_{2k}\cdot e^{-(2\pi k)^2 t}+4\pi k\cdot B_{2k-1}\times$$
$$\times\left\{\frac{1}{(2\pi k)^2}\cdot\left[1-e^{-(2\pi k)^2 t}\right]-t\cdot e^{-(2\pi k)^2 t}\right\}-$$
$$-\frac{1}{\pi k}\cdot\int_0^t\left[1-e^{-(2\pi k)^2(t-\tau)}\right]\cdot\left[\int_0^\tau e^{-(2\pi k)^2(t-\sigma)}\cdot F_{2k-1}(u;\sigma)\right]d\tau+$$
$$+\frac{1}{(2\pi k)^2}\cdot\int_0^t\left[1-e^{-(2\pi k)^2(t-\tau)}\right]\cdot F_{2k}(u;\tau)d\tau,$$
$$(k=1,2,\dots;t\in[0,T])$$

where A_k,B_k $(k=0,1,\dots)$ are arbitrary constants.

From this it is easy to obtain that the functions $u_k(t)$ $(k=0,1,...)$ is reduced to solving the following countable system of nonlinear integro-differential equations:

$$u_0(t)=\phi_0+\psi_0\cdot t+ \\ +\int_0^t\int_0^1(t-\tau)F(u(\tau,x))Y_0(x)dxd\tau\ (t\in[0,T]) \quad (15)$$

$$u_{2k-1}(t)=\phi_{2k-1}+\frac{1}{(2\pi k)^2}\cdot\left[1-e^{-(2\pi k)^2 t}\right]\cdot\psi_{2k-1}+ \\ +\frac{1}{(2\pi k)^2}\cdot\int_0^t\int_0^1\left[1-e^{-(2\pi k)^2(t-\tau)}\right]\cdot F(u(\tau,x))Y_{2k-1}(x)dxd\tau \\ (k=1,2,...;\ t\in[0,T]), \quad (16)$$

$$u_{2k}(t)=\phi_{2k}+\frac{1}{(2\pi k)^2}\cdot\left[1-e^{-(2\pi k)^2 t}\right]\cdot\psi_{2k}- \\ -\frac{1}{\pi k}\cdot\left\{\frac{1}{(2\pi k)^2}\cdot\left[1-e^{-(2\pi k)^2 t}\right]-t\cdot e^{-(2\pi k)^2 t}\right\}\cdot\psi_{2k-1}- \\ -\frac{1}{\pi k}\cdot\int_0^t\left[1-e^{-(2\pi k)^2(t-\tau)}\right]\times \\ \times\left[\int_0^\tau\int_0^1 e^{-(2\pi k)^2(\tau-\sigma)}\cdot F(u(\sigma,x))Y_{2k-1}(x)dxd\sigma\right]d\tau+ \\ +\frac{1}{(2\pi k)^2}\cdot\int_0^t\int_0^1\left[1-e^{-(2\pi k)^2(t-\tau)}\right]\cdot F(u(\tau,x))Y_{2k}(x)dxd\tau \\ (k=1,2,...;\ t\in[0,T]), \quad (17)$$

If in system (15)-(17) instead of functions $Y_k(x)$ $(k=0,1,...)$ we substitute their values determined by relation (6), then system (15)-(17) will take the form:

$$u_0(t)=\phi_0+\psi_0\cdot t+2\int_0^t\int_0^1(t-\tau)F(u(\tau,x))dxd\tau \\ (t\in[0,T]), \quad (18)$$

$$u_{2k-1}(t)=\phi_{2k-1}+\frac{1}{(2\pi k)^2}\cdot\left[1-e^{-(2\pi k)^2 t}\right]\cdot\psi_{2k-1}+ \\ +\frac{1}{\pi^2k^2}\cdot\int_0^t\int_0^1 F(u(\tau,x))\cos 2\pi kx\cdot\left[1-e^{-(2\pi k)^2(t-\tau)}\right] \\ dxd\tau\ (k=1,2,...;\ t\in[0,T]), \quad (19)$$

$$u_{2k}(t)=\phi_{2k}+\frac{1}{(2\pi k)^2}\cdot\left[1-e^{-(2\pi k)^2 t}\right]\cdot\psi_{2k}- \\ -\frac{1}{\pi k}\cdot\left\{\frac{1}{(2\pi k)^2}\cdot\left[1-e^{-(2\pi k)^2 t}\right]-t\cdot e^{-(2\pi k)^2 t}\right\}\cdot\psi_{2k-1}- \\ -\frac{4}{\pi k}\cdot\int_0^1\left[1-e^{-(2\pi k)^2(t-\tau)}\right]\times \\ \times\left[\int_0^\tau\int_0^1 F(u(\sigma,x))\cos 2\pi kx\cdot e^{-(2\pi k)^2(\tau-\sigma)}dxd\sigma\right]d\tau+ \\ +\frac{1}{\pi^2k^2}\cdot\int_0^t\int_0^1(1-x)F(u(\tau,x))\sin 2\pi kx\cdot\left[1-e^{-(2\pi k)^2(t-\tau)}\right]dxd\tau \\ (k=1,2,...;\ t\in[0,T]). \quad (20)$$

Thus, after applying a method similar to the Fourier method, the solution of problem (1)-(3), i.e. finding the function $u(t,x)$, is reduced to solving a system (relative to unknown functions $u_k(t)$ $(k=0,1,...)$ (18)-(20) of nonlinear integro-differential equations, while it is necessary to keep in mind the notation (4) and relations (7), (8).

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