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BASIC METHODS FOR FINDING THE RANGE OF A FUNCTION

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Abstract
A function, as a mathematical model of many real-world situations, allows for describing and studying various dependencies between quantities, thereby facilitating an understanding of the surrounding world. Teaching mathematics is a crucial component of secondary general education and aims to develop logical thinking and mathematical intuition in students, ensuring they acquire skills in solving various practical and interdisciplinary tasks (Tahirov, B.O., Aghazade, S.M., 2024).

Tasks involving finding the range of a function pose significant difficulties for students. Such tasks are consistently included in prestigious mathematical competitions and various Olympiads.

This work is dedicated to the methodology of teaching students to find the range of various functions.

Keywords: *mathematics, function, continuous function, differentiable function, minimum value of a function, inverse function, maximum value of a function, graph of a function, domain of a function, range of a function*

Tasks on finding the range of a function cause considerable difficulties for students. Such tasks are frequently encountered in various mathematical competitions and tests.

The purpose of this article is to explore methods for finding the range of a function. To do this, it is necessary to understand the properties of basic elementary functions, particularly their domain, range, and intervals of monotonicity. When finding the range of a function, properties such as continuity, monotonicity, differentiability, evenness, periodicity, and others are most commonly used.

The most important of these are the following:

1. If a function is $f(x)$ continuous and increasing on a segment $[a;b]$, then its range is a segment. $[f(a);f(b)]$. In this case $A\in[f(a);f(b)]$ for any $f(x)=A$ the equation has a unique solution.

2. If a function is $f(x)$ continuous and decreasing on a segment $[a;b]$, then its range is a segment $[f(b);f(a)]$. In this case $A \in [f(b);f(a)]$ for any $f(x)=A$ the equation has a unique solution.

3. If a function is $f(x)$ continuous on a segment $[a;b]$ and $m=\min_{[a;b]} f(x)$, $M=\max_{[a;b]} f(x)$, then its range is the segment $[m;M]$.

Simple tasks for finding the range of a function can be solved using the following methods:

1. The method of estimating boundaries;
2. Using properties of continuity and monotonicity;
3. Using derivatives;
4. Using the maximum and minimum values of the function;
5. The graphical method;
6. The method of introducing a parameter;
7. The method of inverse functions;
8. Combined methods (Silvestrov V. V., 2004; Shabunin M. I., 2016).

More complex tasks for finding the range of a function can be solved using the following methods:

- By finding the range of complex arguments;
- Through various estimations;
- Using the graph of the function;
- The method of introducing a parameter;
- Finding the domain of the inverse function;
- Using the derivative of the given function;
- Preliminary transformation of the function's form, and so on.

Let us explore the specifics of applying these methods in solving the following problems.

Example 1. Find the range of the function $y=\log_{0,2}\left(5-2\cdot 5^x-25^x\right)$.

We will solve this example using the method of finding the range of complex arguments. First, simplify the logarithmic expression by completing the square of the trinomial:

$$y=\log_{0,2}\left(6-\left(1+2\cdot 5^x+5^{2x}\right)\right)=\log_{0,2}\left(6-\left(5^x+1\right)^2\right).$$

Find the range of the complex arguments of the function sequentially:

$$y=\log_{0,2}\left(6-\left(5^x+1\right)^2\right)$$

$$E\left(5^x\right)=(0;+\infty)\Rightarrow E\left(5^x+1\right)=(1;+\infty)$$

$$E\left(-\left(5^x+1\right)^2\right)=(-\infty;-1)\Rightarrow E\left(6-\left(5^x+1\right)^2\right)=(-\infty;5).$$

Let $t=6-\left(5^x+1\right)^2$. Then the problem reduces to finding the range of the function: $y=\log_{0,2}t$:

$$y=\log_{0,2}t, t\in(-\infty;5).$$

Since the domain of the function $y=\log_{0.2}t$ is the set $(0;+\infty)$, its range on the set $(-\infty;5)$ will be $(0,5)$, i.e. $t\in(0;5)$. Thus, we need to find the range of the function $y=\log_{0,2}t$ on the set $(0,5)$:

When $t\to 0, y\to\infty$ and when $t\to 5, y\to -1$, i.e. $E(y)=(-1;+\infty)$.

Example 2. Find the range of the function $y=\log_2\left(4^x+2^{x+1}-24\right)$.

Solution. By completing the square under the logarithm, we transform the function. Since

$$4^x+2^{x+1}-24=2^{2x}+2\cdot 2^x+1-25=\left(2^x+1\right)^2-25,$$

the given function can be represented as:

$$y=\log_2\left(\left(2^x+1\right)^2-25\right).$$

Now, find the range of its complex arguments:

$$E\left(2^x\right)=(0;+\infty);E\left(2^x+1\right)=(1;+\infty);E\left(\left(2^x+1\right)^2\right)=(1;+\infty);$$

$$E\left(\left(2^x+1\right)^2-25\right)=(-24;+\infty).$$

Thus, $E(f)=(-24;+\infty)$.

Example 3. Find the range of the function

$$y=\cos 5x+6\cos x.$$

This example will be solved using the estimation method. Since the functions $y=\cos 5x$ and $y=\cos x$ are continuous everywhere $-1\le\cos 5x\le 1$ so is $-6\le 6\cos x\le 6$. If we add these inequalities term by term, we get:

$$-7\le\cos 5x+6\cos x\le 7.$$

The most common mistake when finding the range of a function using the estimation method is assuming that if the inequality $A\le f(x)\le B$ holds, the range of the function $f(x)$ must also be $[A;B]$. It is overlooked that this statement is true only when the function $f(x)$ is continuous on the segment $[A;B]$.

Example 4. Find the range of the function

$$y = \log_{0,5}\frac{11x^2}{1+x+x^2}.$$

Solution. Find the range of the function

$$g(x) = \frac{1+x^2}{1+x+x^2} = \frac{1}{1+\frac{x}{x^2+1}}.$$

Consider $\frac{x}{x^2+1}$ and the reciprocal $\frac{x^2+1}{x} = \frac{1}{x}+x$. Obviously, when $x+\frac{1}{x}\geq 2$ and $x>0$и $x+\frac{1}{x}\leq -2$ when $x<0$. Thus,

$$-2 \leq \frac{x^2+1}{x} \leq 2.$$

Then,

$$-\frac{1}{2} \leq \frac{x}{x^2+1} \leq \frac{1}{2} \Leftrightarrow \frac{1}{2} \leq 1+\frac{x}{x^2+1} \leq \frac{3}{2}.$$

From this, we have:

$$\frac{2}{3} \leq \frac{1}{1+\frac{x}{x^2+1}} \leq 2.$$

Thus, the range of the function $g(x)$ is: $E(g) = \left[\frac{2}{3};2\right]$. Therefore, the range of the function is

$$y = \log_{0,5}\frac{1+x^2}{1+x+x^2}$$

since the

$$E(f) = \log_{0,5}(g(x)),$$

outer function is decreasing

$$E(f) = \left[\log_{0,5}2;\log_{0,5}\frac{2}{3}\right] = \left[-1;\log_2\frac{2}{3}\right].$$

Example 5. Find the range of the function

$$f(x) = \cos 4x + 2\cos 2x$$

Solution. Using the double-angle cosine formula $\cos 4x = 2\cos^2 2x - 1$. Therefore, we have:

$$f(x) = 4\cos^2 2x + 2\cos 2x - 1$$

If we let, then $t = \cos 2x$, then

$$f(x) - 4t^2 + 2t - 3, -1 \leq t \leq 1$$

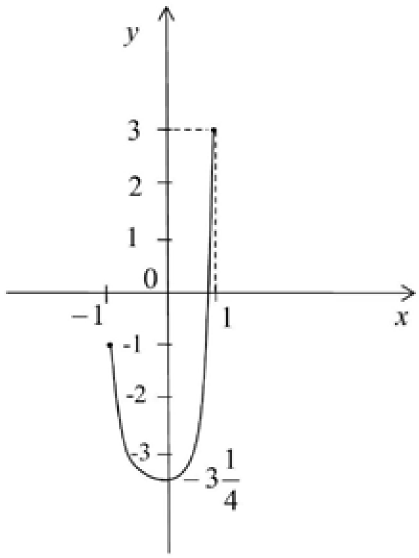
Construct the graph $f(x)$ on the plane $x0y$. First, find the zeros of the function.

$$4t^2+2t-3=0, \frac{D}{4} = 1+12 = 13, x = \frac{-1\pm\sqrt{13}}{4}$$

$$t_1 = \frac{-1-\sqrt{13}}{4} < -1, t_2 = \frac{\sqrt{13}-1}{4} < 1$$

It is easy to find the coordinates of the vertex of a parabola $-B\left(-\frac{1}{4};-3\frac{1}{4}\right)$. In the segment $[-1;1]$ one of the zeros of the function is located $f(t)$: lies in the segment $[-1;1]$. Is $f(1)=3, f(-1)=-1$. Thus, the range of the function f is $E_f = \left[-3\frac{1}{4};3\right]$.

The next example will be solved using the method of introducing an auxiliary parameter.



Example 6. For which values of the parameter does the a equation

$$\sqrt{x+4} = a(x^2+1)$$

have a unique solution $[-3;-1]$ on the segmen?

First, rewrite the original equation in the form

$$\frac{\sqrt{x+4}}{x^2+1} = a, x\in[-4;\infty).$$

For this equation to have at least one root on the segment $[-3;-1]$, the number amust belong to the range of $f(x)=\frac{\sqrt{x+4}}{x^2+1}$ on the segment $[-3;-1]$. Obviously, this function is continuous on $[-4;+\infty)$.

The function $y=x^2+1$ continuous and decreasing on $(-\infty;0]$, so the function $y=\frac{1}{x^2+1}$ is continuous and increasing on $(-\infty;0]$.

The function $y=\sqrt{x+4}$ is continuous and increasing on $[-4;+\infty)$. Then the function is $f(x)=\frac{\sqrt{x+4}}{x^2+1}$ increasing and positive on $[-3;-1]$, as the product of two positive, continuous, $[-3;-1]$ and increasing functions $[f(-3);f(-1)]=\left[\frac{1}{10};\frac{\sqrt{3}}{2}\right]$.

As noted, the solvability of the equation $f(x)$ – a on a certain interval is X equivalent to the a parameter belonging to the range $f(x)$ of on X.

Consequently, the range of the function $f(x)$ on the interval coincides with X the range of the parameter a for which the equation $f(x)=a$ has at least one root.

Example 7. Find the $E(f)$ range of the function $f(x)=\frac{3x^2-7x+4}{x^2+2}$.

This example will be solved using the method of introducing a parameter $E(f)$ according to which coincides with the range of the parameter a for which the equation $\frac{3x^2-7x+4}{x^2+2}=a$ has at least one root, Simplifying the equation, we get:

$$3x^2-7x+4=a\left(x^2+2\right)\Leftrightarrow(3-a)x^2-7x+4-8a=0$$

When $a=3$ the last equation becomes linear with a non-zero coefficient for x.

$$7x=4-8a$$

This equation has a unique solution $x=\frac{1}{7}(4-8a)$ (is any number).

When $a\neq 3$ the equation is quadratic. Therefore, it is solvable if and only if its discriminant

$$D=49-4(3-a)(4-8a)=-32a^2+112a+1\geq 0$$

Solving the resulting inequality, we get:

$$\frac{14-3\sqrt{22}}{8}\leq a\leq\frac{14+3\sqrt{22}}{8}$$

Since the point $a=3$ belongs to the segment $\left[\frac{14-3\sqrt{22}}{8};\frac{14+3\sqrt{22}}{8}\right]$, the desired range of the parameter a, and thus the range of the function $E(f)$ is the entire segment. To find the range of a function, one can consider the method of the inverse function, which involves solving x the equation $f(x)=y$ with respect to, treating as a parameter y. If the equation $f(x)=y$ has multiple solutions, $x=g_1(y), x=g_2(y)$ etc., then is the union of the domains of the functions, $g_1(y)$, $g_2(y)$, etc.

Example 8. Find the range of the function $y=7^{\frac{2}{1-5^x}}$.

This example will be solved using the method of finding the domain of the inverse function.

From the equation $y=7^{\frac{2}{1-5^x}}$, find x in terms of y:

$$\frac{2}{1-5^x}=\log_7 y\Leftrightarrow 5^x=\frac{\log_7 y-2}{\log_7 y}\Leftrightarrow x=\log_5\frac{\log_7 y-2}{\log_7 y}$$

Now, find the domain of the function

$$x=g(y)=\frac{\log_7 y-2}{\log_7 y}\Leftrightarrow\left[\begin{array}{l}\log_7 y>2\\ \log_7 y<2\end{array}\right.\Leftrightarrow\left[\begin{array}{l}y>49\\ 0<y<1\end{array}\right.$$

Thus, the range of the function $y=7^{\frac{2}{1-5^x}}$ is $E(f)=(0;1)\cup(49;+\infty)$.

Example 9. Thus, the range of the function is $f(x)=\cos 2x-3\cos x-1$.

Solution. First, simplify the formula of the function:

$$\cos 2x-3\cos x-1=\cos 2x-3\cos x-2+1=$$
$$=1+\cos 2x-3\cos x-2=2\cos^2 x-3\cos x-2$$

Introduce the variable $t-\cos x$, then

$$f(t)=2t^2-3t-2, t\in[-1;1]$$

This example will be solved using the graphical method. Find the zeros of the function

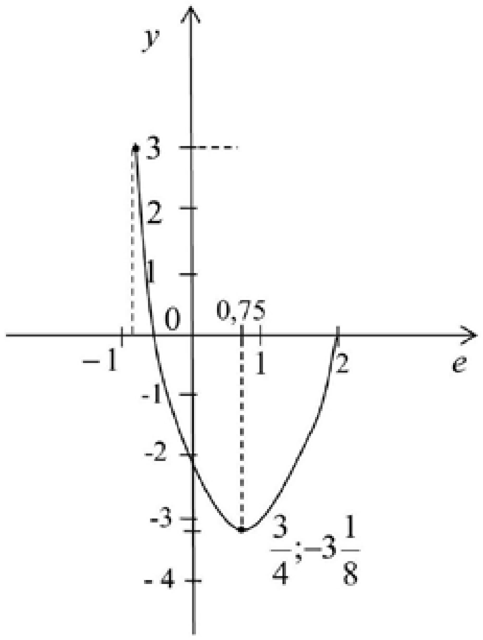
$$2t^2-3t-2=0$$

$D = 9 + 16 = 25;$

$t = \frac{3 \pm 5}{4};$

$t_1 = 2; t_2 = -\frac{1}{2}; t = -\frac{1}{2},$

$t = 2$ is not included in the segment $[-1;1]$.



Now, find the coordinates of the parabola's vertex

$$\left(-\frac{b}{2a}; f\left(-\frac{b}{2a}\right)\right) = \left(\frac{3}{4}; f\left(\frac{3}{4}\right)\right).$$

$$f\left(\frac{3}{4}\right) = 2\cdot\left(\frac{9}{16}\right) - 3\cdot\frac{3}{4} - 2 = \frac{9}{8} - \frac{9}{4} - 2 =$$

$$= \frac{9 - 18 - 16}{8} = -\frac{25}{8}.$$

Thus, the vertex of the parabola is at the point.

$$\left(\frac{3}{4}; -3\frac{1}{8}\right).$$

Now, find the values of the function at the points –1 and 1:

$$f(-1) = 3, f(1) = -3.$$

From the graph of the function, it is clear that

$$E(f) = \left[-3\frac{1}{8}; 3\right].$$

Example 10. Find the range of the function

$$y = \sqrt{36 - x^2}.$$

Solution. Transform the function's expression:

$$y^2 - 36 - x^2 \text{ or } x^2 + y^2 = 36.$$

The latter expression is the equation of a circle with radius 6. Thus, the graph is the upper semicircle of this circle. Therefore, the range of the function is $|x| \le 6, y \ge 0 : E(f) = [0;6]$.

Answer: [0;6].

When finding the range of a function, it is often necessary to use the derivative of the given function defined on a segment.

In this case, the following scheme can be used:

Find the derivative of the given function;

1. Find the critical points of the function and select those that belong to the domain of the function;

2. Calculate the values of the function at the endpoints of the segment and the selected critical points;

3. Among the found values, select the minimum and maximum values;

4. The range of the function is enclosed between these values.

Example 11. Find the range of the function

$$y = \sqrt{25 - x^2}, x \in [-5;5].$$

Solution.

$$f'(x) = \frac{1}{2}\left(25 - x^2\right)^{\frac{1}{2}-1} \cdot \left(25 - x^2\right)' =$$

$$= \frac{1}{2} \cdot \frac{(-2x)}{\sqrt{25 - x^2}} = \frac{-x}{\sqrt{25 - x^2}} = 0,$$

If $x = 0$, then $f'(x) = 0$,

if $x = 5$ and $x = -5$, then does not exist.

We obtain three critical points:

$$x = 0, x = -5, x = 5.$$

If $x = 0$, then $f(0) = 5$;

If $x = -5$, then $f(-5) = 0$;

If $x = 5$, then $f(5) = 0$.

Thus, it is found that the minimum value of the function is 0, and the maximum is 5. Therefore

$$E(f) = [0;5].$$

Let us consider some non-standard techniques for finding the range of a function $f(x)$. One of them is the preliminary transformation of the function.

Example 12. Find the range of the function

$$f(x)=\sqrt{3-x}+\sqrt{3+x}.$$

Since $E\left(2\cdot\sqrt{9-x^2}\right)=[0;6]$, then $f^2(x)=[6;12]$.

From this, we have: $E(f)=\left[\sqrt{6};\quad 2\sqrt{3}\right]$.

Conclusions.

1. From the examples solved above, it follows that finding the range of functions involves a variational approach, for example, between functions that are derived from one another as inverses (Aliyev S. J., Takhirov B. O., Hashimova T. A., 2022).

2. Teaching students to find the range of functions is significant for the following reasons: it develops functional thinking, introduces students to the idea of universal continuity and infinity, and forms skills in analyzing and finding dependencies between changes in various objects and working with abstract material (Takhirov B. O., Namazov F. M., 2024).

3. Tasks on finding the range of a function are included in various competitions and Olympiads. Therefore, it is necessary to study them.

4. The ability to quickly find the range of a quadratic function using various methods will help save time on completing tasks in the future.

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