Section 4. Materials science

DOI:10.29013/EJTNS-26-2-22-24

A STUDY ON A CLASS OF FOURTH ORDER NONLINEAR DIFFERENTIAL EQUATIONS

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Cite: *Aliyev S., Aliyeva A., Aliyev S. (2026). A Study on a Class of Fourth Order Nonlinear Differential Equations. European Journal of Technical and Natural Sciences 2026, No 2.* https://doi.org/10.29013/EJTNS-26-2-22-24

Abstract

This work is dedicated to the study of certain properties of the almost everywhere solution to a one-dimensional mixed problem for one class of fourth order nonlinear equations. The almost everywhere solution of mixed problem under consideration is sought in the form of Fourier series. After applying Fourier method, the problem of finding unknown Fourier coefficients $u_n(t)\ (n=1,2,...)$ of sought almost everywhere solution $u(t,x)$ is reduced to solving some countable system of nonlinear integral equations.

Keywords: *fourth order nonlinear equation, one-dimensional mixed problem, almost everywhere solution, Fourier method*

The study examines certain properties of the almost everywhere solution to the following one-dimensional mixed problem:

$$\begin{cases} u_{txx}(t,x) - \alpha\, u_{xxxx}(t,x) = F(t,x,u(t,x),u_x(t,x),u_{xx}(t,x),u_{xxx}(t,x)) \\ \qquad\qquad (0 \le t \le T,\ 0 \le x \le \pi), & (1) \\ u(0,x) = \phi(x)\ (0 \le x \le \pi), & (2) \\ u(t,0) = u(t,\pi) = u_{xx}(t,0) = u_{xx}(t,\pi) = 0 \quad (0 \le t \le T), & (3) \end{cases}$$

where $\alpha > 0$ is a fixed number, $0 < T < +\infty$; F and ϕ are given functions, and $u(t,x)$ is a sought function.

Definition

We define an almost everywhere solution of problem (1)-(3) as a function $u(t,x)$ with the following properties:

a) $u(t,x)$, $u_x(t,x)$, $u_{xx}(t,x)$, $u_{xxx}(t,x)$, $u_t(t,x)$, $u_{tx}(t,x) \in C([0,T]\times[0,\pi])$; $u_{xxxx}(t,x)$, $u_{txx}(t,x) \in C([0,T];L_2(0,\pi))$;

b) equation (1) is satisfied almost everywhere in $[0,T]\times[0,\pi]$;

c) all the conditions (2) and (3) are satisfied in ordinary sense.

Previous studies have addressed various aspects of mixed problems for nonlinear equations. In particular, the paper by (Aliyev S., Heydarova M., and Aliyeva A., 2024), the local (in small) existence of a classical solution to the mixed problem under consideration was established.

It should also be mentioned that some methodological ideas used in the present study are based on the approaches developed in the works of (Davis P. L., 1971), (Aliyev S., Aliyeva A., and Abdullayeva G., 2019),(Khudaverdiyev K. I. and Sadikhov M. N., 2008), (Khudaverdiyev K. I., Heydarova M.N, 2008), (Azizbayov, E.I.,2019) and (Azizbayov, E.I., 2022).

As the system $\{\sin nx\}_{n=1}^{\infty}$ forms a basis in the space $L_2(0,\pi)$, then it is obvious that every almost everywhere solution $u(t,x)$ of problem (1)-(3) has the following form:

$$u(t,x)=\sum_{n=1}^{\infty}u_n(t)\sin nx\,, \tag{4}$$

where

$$u_n(t)=\frac{2}{\pi}\int_0^{\pi}u(t,x)\sin nxdx\,. \tag{5}$$
$$(n=1,2,...;\ t\in[0,T])$$

In the next, after applying Fourier method, the finding of functions $u_n(t)$ $(n=1,2,...)$ is reduced to solving the following countable system of nonlinear integral equations:

$$u_n(t)=\phi_n\cdot e^{-\alpha n^2 t}-\frac{2}{\pi n^2}\times$$
$$\times\int_0^t\int_0^{\pi}\mathrm{P}(u(\tau,x))\sin nx\cdot e^{-\alpha n^2(t-\tau)}dxd\tau \tag{6}$$
$$(n=1,2,...;\ t\in[0,T]).$$

where

$$\phi_n\equiv\frac{2}{\pi}\int_0^{\pi}\phi(x)\sin nxdx\ \ (n=1,2,...), \tag{7}$$

$$\mathrm{P}(u(t,x))\equiv F(t,x,u(t,x),u_x(t,x),u_{xx}(t,x),u_{xxx}(t,x))\,, \tag{8}$$

with notations (4) and (5) taken into account.

Using the definition of almost everywhere solution of problem (1)-(3), it is to prove the following

Theorem

If $u(t,x)=\sum_{n=1}^{\infty}u_n(t)\sin nx$ is any almost everywhere solution of problem (1)-(3), then functions $u_n(t)$ $(n=1,2,...)$ satisfy on the $[0,T]$ system (6).

Proof

Let $u(t,x)=\sum_{n=1}^{\infty}u_n(t)\sin nx$ be any almost-everywhere solution of problem (1)-(3). Then, obviously, the functions $u_n(t)$ $(n=1,2,...)$ are defined by relation (5). First, we adopt the notation

$$\mathrm{P}_n(u;t)\equiv\frac{2}{\pi}\int_0^{\pi}P(u(t,x))\sin nxdx \tag{9}$$
$$(n=1,2,...;\ t\in[0,T]),$$

and the operator P is defined by relation (8).

Next, multiplying both parts of equation (1) by the function $\frac{2}{\pi}\sin nx$, integrating the resulting equality over x from 0 to π and using the notation (9), we obtain that for any fixed $(n=1,2,...)$ for any $t\in[0,T]$:

$$\frac{2}{\pi}\int_0^{\pi}u_{txx}(t,x)\sin nxdx=\frac{2}{\pi}\{u_{tx}(t,x)\cdot\sin nx\}_{x=0}^{x=\pi}-$$
$$\frac{2}{\pi}n\int_0^{\pi}u_{tx}(t,x)\cos nxdx=-\frac{2n}{\pi}\int_0^{\pi}u_{tx}(t,x)\cos nxdx=$$
$$=-\frac{2n}{\pi}\{u_t(t,x)\cdot\cos nx\}_{x=0}^{x=\pi}-\frac{2n}{\pi}n\int_0^{\pi}u_t(t,x)\sin nxdx=$$
$$=-\frac{2n^2}{\pi}\int_0^{\pi}u_t(t,x)\sin nxdx= \tag{10}$$
$$=-n^2\cdot\frac{d}{dt}\left(\frac{2}{\pi}\int_0^{\pi}u(t,x)\sin nxdx=\right)=n^2\cdot u_n'(t);$$

$$\frac{2}{\pi}\int_0^{\pi}u_{xxxx}(t,x)\sin nxdx=\frac{2}{\pi}\{u_{xxx}(t,x)\cdot\sin nx\}_{x=0}^{x=\pi}-$$

$$-\frac{2}{\pi}\cdot n\cdot\int_0^{\pi}u_{xxx}(t,x)\cos nxdx=-\frac{2n}{\pi}\cdot\left\{u_{xx}(t,x)\cdot\cos nx\right\}_{x=0}^{x=\pi}-$$

$$-\frac{2n}{\pi}\cdot n\cdot\int_0^{\pi}u_{xx}(t,x)\sin nxdx=-\frac{2n^2}{\pi}\cdot\int_0^{\pi}u_{xx}(t,x)\sin nxdx=$$

$$=-\frac{2n}{\pi}\cdot\left\{u_x(t,x)\cdot\sin nx\right\}_{x=0}^{x=\pi}+\frac{2n^2}{\pi}\cdot n\cdot\int_0^{\pi}u_x(t,x)\cos nxdx=$$

$$=\frac{2n^3}{\pi}\cdot\int_0^{\pi}u_x(t,x)\cos nxdx=\frac{2n^3}{\pi}\cdot\left\{u_x(t,x)\cdot\cos nx\right\}_{x=0}^{x=\pi}+$$

$$+\frac{2n^3}{\pi}\cdot n\cdot\int_0^{\pi}u(t,x)\sin nxdx=\frac{2n^4}{\pi}\cdot\int_0^{\pi}u(t,x)\sin nxdx=$$

$$=n^4\cdot\left(\frac{2}{\pi}\int_0^{\pi}u(t,x)\sin nxdx=\right)=n^4\cdot u_n(t). \quad (11)$$

Due to (10) – (11), relation (9) takes the form:

$$-n^2u_n'(\tau)-\alpha n^4\cdot u_n(t)+P_n(u;t),$$

hence

$$u_n'(t)+\alpha n^2u_n(t)=$$
$$=-\frac{1}{n^2}\cdot P_n(u;t)\ \ (n=1,2,...;t\in[0,T]) \quad (12)$$

And from the initial condition (2) it follows that

$$u_n(0)=\frac{2}{\pi}\int_0^{\pi}u(0,x)\sin nxdx=$$
$$=\frac{2}{\pi}\int_0^{\pi}\phi(x)\sin nxdx=\phi_n\ \ (n=1,2,...) \quad (13)$$

Thus, the functions $u_n(t)\ \ (n=1,2,...)$ are the solution of the Cauchy problem (12), (13).

Easy to show the problem of (12), (13) is equivalent to system (6). The theorem is proved.

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submitted 14.03.2026;
accepted for publication 28.03.2026;
published 31.03.2026

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