Section 1. Information technology

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THEORETICAL BASES OF FUZZY POLYNOMIAL EQUATIONS

Abstract. In the article the formal theory of equations based on fuzzy matches is proposed. The necessary and sufficient conditions for the existence of solutions of inverse problems - problems of fuzzy diagnostics are given.

Keywords: equations with fuzzy matches, inverse problems.

Introduction

The present stage of evolution of artificial intelligence systems is defined by rapid implementation of decision support systems with knowledge bases and inference machines, based on empirical and hard-to-classify knowledge of experts in various subject areas.

One of the key areas in the research and synthesis of knowledge-based systems is the problem of developing expert fuzzy diagnostic systems, whose theoretical basis is the concepts of fuzzy and linguistic variables and fuzzy relations, a generalization of which are fuzzy matches.

Therefore, the purpose of this study is to create a formal theory of fuzzy equations that constitute the conceptual basis for the synthesis of intelligent systems of fuzzy diagnostics.

Key concepts and definitions

A fuzzy matching $\tilde{A}(A,B)=\tilde{A}(A\times B)$ is a fuzzy subset of the Cartesian product where A and B re non-empty (crisp) sets..

Let A×B and B×C be given fuzzy matches $\tilde{A}$ and $\tilde{X}$:

$$\tilde{A}=\iint\limits_{A\times B}\frac{\mu_{\tilde{A}}(a,b)}{(a,b)};\ \tilde{X}=\iint\limits_{B\times C}\frac{\mu_{\tilde{X}}(b,c)}{(b,c)}.$$

A composition of fuzzy matches is a fuzzy matching in the following form:

$$\tilde{A}\circ\tilde{X}=\iint_{A\times C}\frac{\mu_{\tilde{A}\circ\tilde{X}}(a,c)}{(a,c)},$$

defined at A×C by a membership function:

$$\mu_{\tilde{A}\circ\tilde{X}}(a,c)=\left[\begin{matrix}\sup T\\ \inf I\end{matrix}\right._{b\in B}\left(\mu_{\tilde{A}}(a,b),\mu_{\tilde{X}}(b,c)\right), \quad (1)$$

where T is t-norm, I is the induced implicator.

A fuzzy (relational [1]) equation is the following equation:

$$\tilde{A}\circ\tilde{X}=\tilde{Y}, \quad (2)$$

where $\tilde{A}(A,B),\tilde{X}(B,C),\tilde{Y}(A,C)$ are fuzzy matches. Or in matrix notation: $A\circ X=Y$.

A direct problem for equation (2) is the problem of finding a fuzzy correspondence Y given A, X and the composition rule. In this case $Y=A\circ X$ and the solution of the direct problem is trivial and can be found by (1).

The inverse (right) problem for equation (2) is the problem of finding a representation of a fuzzy match X given A, Y the composition rule. The inverse problem is fundamentally complex (can have interval solutions or no solutions at all).

The simplest fuzzy equation is the following equation:

$$\left[\begin{matrix}T\\ \cdots\\ I\end{matrix}\right](a,x)=y.$$

where $a, x, y\in[0,1]$.

A polynomial fuzzy equation is the following equation: $a\circ x=y$, where

$$a=\begin{pmatrix}a_1 & a_2 & \cdots & a_n\end{pmatrix}, x=\begin{pmatrix}x_1\\ x_2\\ \vdots\\ x_n\end{pmatrix}; a_i,x_i\left(i=\overline{1,n}\right), y\in[0,1]$$

A polynomial equation system is the following fuzzy equation:

$$\begin{pmatrix}a_{11} & a_{12} & \cdots & a_{1p}\\ a_{21} & a_{22} & \cdots & a_{2p}\\ \vdots & \vdots & \ddots & \vdots\\ a_{m1} & a_{m2} & \cdots & a_{mp}\end{pmatrix}\circ\begin{pmatrix}x_1\\ x_2\\ \vdots\\ x_p\end{pmatrix}=\begin{pmatrix}y_1\\ y_2\\ \vdots\\ y_m\end{pmatrix}.$$

Main subject

1. Analysis of the simplest equations

Let us examine equations with triangular norms:

$$T(a,x)=y. \quad (3)$$

Assertion. For the equation (3) to have a solution x^0 it is necessary and sufficient to satisfy the condition $a\geq y$ (the proof is trivial).

For example, the equation $T(0.5,x)=0.8$ has no solutions.

For an equation with implicators:

$$I(a,x)=y. \quad (4)$$

Assertion. For the equation (4) to have a solution x^0 it is necessary and sufficient to satisfy the condition $1-a\leq y$ (the proof is trivial).

For example, the equation $I(0.5,x)=0.3$ has no solutions.

2. Analysis of polynomial equations

Let us examine the following equation:

$$\max_{i=\overline{1,n}}\left(T(a_i,x_i)\right)=y. \quad (5)$$

Assertion. For the equation (5) to have a solution

$$x^0=\begin{pmatrix}x_1^0\\ x_2^0\\ \vdots\\ x_n^0\end{pmatrix}$$

it is necessary and sufficient to satisfy the condition:

$$\exists j\,(1\leq j\leq n): a_j\geq y.$$

For example, the equation $\begin{pmatrix}0.2 & 0.4 & 0.5 & 0.6\end{pmatrix}\circ\begin{pmatrix}x_1\\ x_2\\ x_3\\ x_4\end{pmatrix}=0.7$ has no solutions.

Proof.

Necessity.

Let x^0 – be the solution of the problem (5). This means that there exists at least one number $1 \le j \le n$, for which the equation $T\left(a_j, x_j^0\right) = y$, is satisfied, and this means that $a_j \ge y$ (see solution of simple equations).

It has been proven that $\exists x^0 = \begin{pmatrix} x_1^0 \\ x_2^0 \\ \vdots \\ x_n^0 \end{pmatrix} \Rightarrow \underset{[1,n]}{\exists\, j} : a_j \ge y.$

Sufficiency.

Let $\underset{[1,n]}{\exists\, j} : a_j \ge y$, then at least one of the equations $T\,(a_j, x_j) = y$ has a solution, then (for (5)):

$$\max_i \left(T\left(a_i, x_i\right)\right) = \max\left(T\left(a_1, x_1\right) \le y, T\left(a_2, x_2\right) \le y, \ldots, T\left(a_j, x_j\right) = y, \ldots\right) = y.$$

It has been proved that $\underset{[1,n]}{\exists\, j} : a_j \ge y \Rightarrow \exists \begin{pmatrix} x_1^0 \\ x_2^0 \\ \vdots \\ x_n^0 \end{pmatrix}$.

Let $u = \left[\begin{array}{l:l} y & T_m \\ y/a_i & T_p \\ y\text{-}a_i + 1 & T_w \end{array}\right.$, then the components of the maximal solution are found as follows:

$$\bar{x}_i^0 = \left[\begin{array}{l:l} 1 & a_i \le y \\ u & a_i > y \end{array}\right. \left(i = \overline{1,n}\right),$$

Where *Tm*, *Tp* and *Tw* are the logical, algebraic and drastic products.

Let us consider $Q = \{i : a_i \ge y\}$.

The components of the minimal solutions are:

$$\underline{x}_i^0(k) = \left[\begin{array}{l} \begin{array}{l:l} 0 & a_i < y \end{array} \\ \left[\begin{array}{l:l} u & i \in Q \\ 0 & i \notin Q \end{array}\right. \end{array}\right. \left(k = 1, |Q|, i = \overline{1,n}\right).$$

Let us analyse the equations with implicators:

$$\min_{i=1,n}\left(I\left(a_i, x_i\right)\right) = y \qquad (6)$$

Assertion. For equation (6) to have a solution

$$x^0 = \begin{pmatrix} x_1^0 \\ x_2^0 \\ \vdots \\ x_n^0 \end{pmatrix},$$

it is necessary and sufficient to satisfy the condition:

$$\exists j\left(1 \le j \le n\right) : a_j \ge 1 - y.$$

For example, the equation $(0.2 \quad 0.4 \quad 0.5 \quad 0.6) \circ$ $\circ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 0.3$ For example, the equation

Proof.

Necessity. Let us consider x^0 to be the solution of the problem (6). This means that there exists at least one number $1 \le j \le n$, for which the equation $I\left(a_j, x_j^0\right) = y$, is satisfied, and this means that $a_j \ge 1 - y$ (see solution of simple equations).

Therefore, the following is true:

$$\exists x^0 = \begin{pmatrix} x_1^0 \\ x_2^0 \\ \vdots \\ x_n^0 \end{pmatrix} \Rightarrow \underset{[1,n]}{\exists\, j} : a_j \ge 1 - y.$$

Sufficiency.

Let $\underset{[1,n]}{\exists\, j} : a_j \ge 1 - y$, then at least one of the equations $I\left(a_j, x_j^0\right) = y$ has a solution, therefore (for (6)):

$$\min_i\left(I\left(a_i, x_i\right)\right) = \min\left(I\left(a_1, x_1\right) \ge y, I\left(a_2, x_2\right) \ge y, \ldots, I\left(a_j, x_j\right) = y, \ldots\right) = y$$

Therefore, the following is true:

Let $u=\left[\begin{array}{c:c} y & I_m \\ \dfrac{a_i+y-1}{a} & I_p \\ a_i+y-1 & I_w \end{array}\right.$, then the components of the minimal solution will be found as follows:

$$\underline{x}_i^0=\left[\begin{array}{c:c} 0 & a_i \le 1-y \\ u & a_i > 1-y \end{array}\right.\left(i=\overline{1,n}\right).$$

Here *Im*, *Ip* and *Iw* are the implicators induced by the norms Tm, Tp and Tw using a standard inverter.

Let $Q=\{i : a_i \ge 1-y\}$.

The components of the maximal solutions:

$$\bar{x}_i^0(k)=\left[\begin{array}{c} \begin{array}{c:c} 1 & a_i < 1-y \end{array} \\ \hdashline \left[\begin{array}{c:c} u & i\in Q \\ \hdashline 1 & i\notin Q \end{array}\right. \end{array}\right.\left(k=1,|Q|,i=\overline{1,n}\right).$$

3.Principles for solving systems of polynomial equations

Inverse problems for systems of polynomial equations are solved according to the following:

$$\left[\begin{array}{l} \left\{\max\limits_k\left(T\left(a_{ik},x_k\right)\right)=y_i\right. \\ \left\{\min\limits_k\left(I\left(a_{ik},x_k\right)\right)=y_i\right. \end{array}\right.\left(i=\overline{1,m}\right)$$

For example, the well-known (classical) fuzzy car fault diagnosis problem [2] is formulated as follows:

$$\begin{pmatrix} 1.0 & 0.8 & 0.7 & 1.0 \\ 0.1 & 0.9 & 0.8 & 0.5 \\ 0.2 & 1.0 & 0.5 & 0.2 \end{pmatrix}\circ\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}=\begin{pmatrix} 0.9 \\ 0.1 \\ 0.2 \end{pmatrix}$$

The equation system using the max-min composition in this case is as follows:

$$\begin{cases} \max\left(\min\left(1.0,x_1\right),\min\left(0.8,x_2\right),\min\left(0.7,x_3\right),\min\left(1.0,x_4\right)\right)=0.9 \\ \max\left(\min\left(0.1,x_1\right),\min\left(0.9,x_2\right),\min\left(0.8,x_3\right),\min\left(0.5,x_4\right)\right)=0.1 \\ \max\left(\min\left(0.2,x_1\right),\min\left(1.0,x_2\right),\min\left(0.5,x_3\right),\min\left(0.2,x_4\right)\right)=0.2 \end{cases}.$$

The table shows the solutions to the three equations and the system as a whole.

Table 1.– Solution to the diagnostic problem

i	x^0	$\bar{x}^0$	$\underline{x}^0$
1.	$\left[\begin{array}{c:c:c:c} 0\le x_1^0<0.9 & 0\le x_2^0\le 1 & 0\le x_3^0\le 1 & x_4^0=0.9 \\ \hdashline x_1^0=0.9 & 0\le x_2^0\le 1 & 0\le x_3^0\le 1 & 0\le x_4^0\le 0.9 \end{array}\right.$	$(0.9 \vdots 1 \vdots 1 \vdots 0.9)$	$\left(\left[\begin{array}{c:c:c:c} 0 & 0 & 0 & 0.9 \\ \hdashline 0.9 & 0 & 0 & 0 \end{array}\right.\right)$
2.	$\left[\begin{array}{c} \begin{array}{c:c} 0\le x_1^0<0.1 & \left[\begin{array}{c} \begin{array}{c:c} 0\le x_2^0<0.1 & \left[\begin{array}{c:c} 0\le x_3^0<0.1 & x_4^0=0.1 \\ \hdashline x_3^0=0.1 & 0\le x_4^0\le 0.1 \end{array}\right. \end{array} \\ \hdashline \begin{array}{c:c:c} x_2^0=0.1 & 0\le x_3^0\le 0.1 & 0\le x_4^0\le 0.1 \end{array} \end{array}\right. \end{array} \\ \hdashline \begin{array}{c:c:c:c} 0.1\le x_1^0\le 1 & 0\le x_2^0\le 0.1 & 0\le x_3^0\le 0.1 & 0\le x_4^0\le 0.1 \end{array} \end{array}\right.$	$(1 \vdots 0.1 \vdots 0.1 \vdots 0.1)$	$\left(\left[\begin{array}{c:c:c:c} 0 & 0 & 0 & 0.1 \\ \hdashline 0 & 0 & 0.1 & 0 \\ \hdashline 0 & 0.1 & 0 & 0 \\ \hdashline 0.1 & 0 & 0 & 0 \end{array}\right.\right)$
3.	$\left[\begin{array}{c} \begin{array}{c:c} 0\le x_1^0<0.2 & \left[\begin{array}{c} \begin{array}{c:c} 0\le x_2^0<0.2 & \left[\begin{array}{c:c} 0\le x_3^0<0.2 & 0.2\le x_4^0\le 1 \\ \hdashline x_3^0=0.2 & 0\le x_4^0\le 1 \end{array}\right. \end{array} \\ \hdashline \begin{array}{c:c:c} x_2^0=0.2 & 0\le x_3^0\le 0.2 & 0\le x_4^0\le 1 \end{array} \end{array}\right. \end{array} \\ \hdashline \begin{array}{c:c:c:c} 0.2\le x_1^0\le 1 & 0\le x_2^0\le 0.2 & 0\le x_3^0\le 0.2 & 0\le x_4^0\le 1 \end{array} \end{array}\right.$	$(1 \vdots 0.2 \vdots 0.2 \vdots 1)$	$\left(\left[\begin{array}{c:c:c:c} 0 & 0 & 0 & 0.2 \\ \hdashline 0 & 0 & 0.2 & 0 \\ \hdashline 0 & 0.2 & 0 & 0 \\ \hdashline 0.2 & 0 & 0 & 0 \end{array}\right.\right)$
$\{\vdots$	$x_1^0=0.9 \vdots 0\le x_2^0\le 0.1 \vdots 0\le x_3^0\le 0.1 \vdots 0\le x_4^0\le 0.1$	$(0.9 \vdots 0.1 \vdots 0.1 \vdots 0.1)$	$(0.9 \vdots 0 \vdots 0 \vdots 0)$

In this case, the maximum solution of the system $(0.9 \vdots 0.1 \vdots 0.1 \vdots 0.1)$ is equal to the intersection of three partial maximum solutions $(0.9 \vdots 1 \vdots 1 \vdots 0.9) \cap (1 \vdots 0.1 \vdots 0.1 \vdots 0.1) \cap (1 \vdots 0.2 \vdots 0.2 \vdots 1)$ (the system is joint).

The semantic interpretation of the resulting solution is presented quite fully in [2].

Conclusion

Theoretical bases of the formal formulation and methodology for finding solutions of fuzzy polynomial equations have been developed. The influence of compositions of fuzzy matches with standard triangular norms and implicators induced on their basis on the existence criteria and components of the synthesized solutions is investigated.

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