Section 1. Materials Science

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THE STUDY OF MIXED PROBLEM FOR ONE CLASS FOURTH ORDER DIFFERENTIAL EQUATIONS

Abstract. In this paper, we study the almost everywhere solution of one dimensional mixed problem for one class fourth order differential equations and some a priori estimates are obtained for the almost everywhere solution of the mixed problem under consideration.

Keywords: mixed problem, differential equation, a priori estimate.

In this work, we study the almost everywhere solution of the following one dimensional mixed problem:

$$\begin{cases} u_{txx}(t,x)-\alpha\, u_{xxxx}(t,x)=F(t,x,u(t,x),u_x(t,x),u_{xx}(t,x),u_{xxx}(t,x))(0\le t\le T,\ 0\le x\le\pi), & (1)\\ u(0,x)=\phi(x)\ (0\le x\le\pi), & (2)\\ u(t,0)=u(t,\pi)=u_{xx}(t,0)=u_{xx}(t,\pi)=0\ \ (0\le t\le T), & (3)\end{cases}$$

where $\alpha>0$ is a fixed number; $0<T<+\infty$; F and ϕ are the given functions, and $u(t,x)$ is a sought function, and under the almost everywhere solution of problem (1)–(3) we understand the following:

$a)$ $u(t,x),u_x(t,x),u_{xx}(t,x),u_{xxx}(t,x),$
$u_t(t,x),u_{tx}(t,x)\in C\big([0,T]\times[0,\pi]\big);$

$u_{xxxx}(t,x),u_{txx}(t,x)\in C\big([0,T];L_2(0,\pi)\big);$

$b)$ equation (1) is satisfied almost everywhere in $(0,T)\times(0,\pi)$;

$c)$ all the conditions (2) and (3) are satisfied in ordinary sense.

There have been many works devoted to the study of mixed problems for nonlinear fourth order (see [1; 2; 3; 5] and references therein).

As the system $\{\sin nx\}_{n=1}^{\infty}$ forms a basis in the space $L_2(0,\pi)$, then it is obvious that every almost everywhere solution $u(t,x)$ of problem (1)–(3) has the following form:

$$u(t,x)=\sum_{n=1}^{\infty}u_n(t)\sin nx\,, \tag{4}$$

where

$$u_n(t)=\frac{2}{\pi}\int_0^{\pi}u(t,x)\sin nx dx\ \ (n=1,2,...;\ t\in[0,T]). \tag{5}$$

In the next, after applying Fourier method, the finding of functions $u_n(t)$ $(n=1,2,...)$ is reduced to solving the following countable system of nonlinear integral equations:

$$u_n(t) = \phi_n \cdot e^{-\alpha n^2 t} - \frac{2}{\pi n^2} \cdot$$
$$\cdot \int_0^t \int_0^\pi \Phi(u(\tau,x)) \sin nx \cdot e^{-\alpha n^2 (t-\tau)} dx d\tau$$
$$(n = 1,2,...;\ t \in [0,T]), \tag{6}$$

where

$$\phi_n \equiv \frac{2}{\pi} \int_0^\pi \phi(x) \sin nx dx \ \ (n = 1,2,...), \tag{7}$$

$$\Phi(u(t,x)) \equiv F(t,x,u(t,x),u_x(t,x), u_{xx}(t,x),u_{xxx}(t,x)), \tag{8}$$

Using the definition of almost everywhere solution of problem (1)–(3), it is easy to prove (see [3]) the following

Lemma. If $u(t,x) = \sum_{n=1}^{\infty} u_n(t) \sin nx$ is any almost everywhere solution of problem (1)–(3), then functions $u_n(t)$ $(n = 1,2,...)$ satisfy the system (6).

We denote by $B^{\alpha}_{\beta,T}$ the set of all functions $u(t,x)$ of the form (4) such that $u_n(t) \in C[0,T]$ and $\sum_{n=1}^{\infty} \left(n^{\alpha} \cdot \max_{0 \le t \le T} |u_n(t)| \right)^{\beta} < +\infty$, where $\alpha \ge 0, 1 \le \beta \le 2$. We define the norm in this set as follows:

$$\|u\|_{B^{\alpha}_{\beta,T}} = \left\{ \sum_{n=1}^{\infty} \left(n^{\alpha} \cdot \max_{0 \le t \le T} |u_n(t)| \right)^{\beta} \right\}^{\frac{1}{\beta}}. \tag{9}$$

It is known (see [5]) that all these spaces are Banach spaces.

Theorem.

1. Let the right side of equation (1) be as follows:

$$F(t,x,u,u_x,u_{xx},u_{xxx}) =$$
$$= f_0(t,u_{xx})) \cdot u_{xxx} + f(t,x,u,u_x,u_{xx},u_{xxx}), \tag{10}$$

where

$a)$ $f_0(t,V) \in C([0,T] \times (-\infty,\infty))$; (11)

$b)$ $f(t,x,u_1,...,u_4) \in C\left([0,T] \times [0,\pi] \times (-\infty,\infty)^4\right)$

and in $[0,T] \times [0,\pi] \times (-\infty,\infty)^4$

$$f(t,x,u_1,...,u_4) \cdot u_3 \le C \cdot$$
$$\cdot (1 + u_1^2 + u_2^2 + u_3^2) + \delta \cdot u_4^2,\ 0 < \delta < \alpha, \tag{12}$$

where $C > 0$ is a constant and $\alpha > 0$ is a number appearing in the equation (1).

2. $\forall R > 0$ in $[0,T] \times [0,\pi] \times [-R,R]^2 \times (-\infty,\infty)^2$

$$|F(t,x,u_1,...,u_4)| \le C_R \cdot (1 + |u_3|^3 + |u_3| \cdot |u_4| + |u_4|), \tag{13}$$

where $C_R > 0$ is a constant.

Then the following a priori estimate holds for all the possible almost everywhere solutions $u(t,x)$ of problem (1)–(3):

$$\|u(t,x)\|_{B^3_{2,T}} \le C, \tag{14}$$

where $C > 0$ is a constant.

Proof. Let $u(t,x) = \sum_{n=1}^{\infty} u_n(t) \sin nx$ be any almost everywhere solution of problem (1)–(3). Then, by virtue of above lemma, functions $u_n(t)$ $(n = 1,2,...)$ satisfy the system (6).

From system (6) we obtain $\forall t \in [0,T]$:

$$\|u\|^2_{B^3_{2,t}} \le a_0 + \frac{2}{\alpha \pi} \cdot \int_0^t \int_0^\pi \{\Phi(u(\tau,x))\}^2 dx d\tau, \tag{15}$$

where

$$a_0 \equiv 2 \sum_{n=1}^{\infty} (n^3 \cdot \phi_n)^2. \tag{16}$$

Then, using conditions of this theorem for $R = R_0$, we obtain from (15) that $\forall t \in [0,T]$:

$$\|u\|^2_{B^3_{2,t}} \le a_0 + \frac{2}{\alpha \pi} \cdot C^2_{R_0} \cdot 4 \int_0^t \int_0^\pi \Big\{ 1 + u^6_{xx}(\tau,x) + $$
$$+ u^2_{xx}(\tau,x) \cdot u^2_{xxx}(\tau,x) + u^2_{xxx}(\tau,x) \Big\} dx d\tau. \tag{17}$$

Now, using following estimates from [3]

$$\|u_{xx}(t,x)\|^2_{C([0,\pi])} \le \frac{\pi^2}{2} \cdot \|u\|^2_{B^3_{2,t}}, \tag{18}$$
$$\int_0^\pi u^2_{xxx}(\tau,x) dx \le \frac{\pi}{2} \cdot \|u\|^2_{B^3_{2,\tau}},$$
$$\int_0^\pi u^2_{xx}(t,x) dx \le C_0,\ \forall t \in [0,T],$$

we obtain $\forall t \in [0,T]$:

$$\int_0^\pi u^6_{xx}(\tau,x) dx \le \|u_{xx}(\tau,x)\|^2_{C([0,\pi])} \cdot \|u_{xx}(\tau,x)\|^2_{C([0,\pi])} \cdot \int_0^\pi u^2_{xx}(\tau,x) dx \le$$
$$\le \pi \int_0^\pi u^2_{xxx}(\tau,x) dx \cdot \frac{\pi^2}{2} \|u\|^2_{B^3_{2,\tau}} \cdot C_0 = \frac{\pi^3}{2} \cdot C_0 \cdot \int_0^\pi u^2_{xxx}(\tau,x) dx \cdot \|u\|^2_{B^3_{2,\tau}}, \tag{19}$$

$$\int_0^\pi u_{xx}^2(\tau,x)\cdot u_{xxx}^2(\tau,x)dx \le \|u_{xx}(\tau,x)\|_{C([0,\pi])}^2 \cdot \int_0^\pi u_{xxx}^2(\tau,x)dx \le$$

$$\le \frac{\pi}{2}\cdot\|u\|_{B_{2,\tau}^3}^2 \cdot \int_0^\pi u_{xxx}^2(\tau,x)dx = \frac{\pi^2}{2}\cdot\int_0^\pi u_{xxx}^2(\tau,x)dx\cdot\|u\|_{B_{2,\tau}^3}^2, \quad (20)$$

$$\int_0^\pi u_{xxx}^2(\tau,x)dx \le \frac{\pi}{2}\cdot\|u\|_{B_{2,\tau}^3}^2. \quad (21)$$

Then, using estimates (19)–(21), from (17) we obtain that $\forall t\in[0,T]$:

$$\|u\|_{B_{2,t}^3}^2 \le a_0 + \frac{8T}{\alpha}\cdot C_{R_0}^2 + \frac{8}{\alpha\pi}\cdot C_{R_0}^2\cdot\int_0^t\left\{\left(\frac{\pi^3}{2}\cdot C_0+\frac{\pi^2}{2}\right)\cdot\int_0^\pi u_{xxx}^2(\tau,x)dx+\frac{\pi}{2}\right\}\cdot\|u\|_{B_{2,\tau}^3}^2\,d\tau. \quad (22)$$

Applying Bellman's inequality [4, p.188, 189], from (22) we obtain:

$$\|u\|_{B_{2,T}^3}^2 \le \left(a_0+\frac{8T}{\alpha}\cdot C_{R_0}^2\right)\cdot\exp\left\{\frac{8}{\alpha\pi}\cdot C_{R_0}^2\cdot\left[\left(\frac{\pi^3}{2}\cdot C_0+\frac{\pi^2}{2}\right)\cdot\int_0^T\int_0^\pi u_{xxx}^2(\tau,x)dxd\tau+\frac{\pi}{2}\cdot T\right]\right\}\le$$

$$\le\left(a_0+\frac{8T}{\alpha}\cdot C_{R_0}^2\right)\cdot\exp\left\{\frac{8}{\alpha\pi}\cdot C_{R_0}^2\cdot\left[\left(\frac{\pi^3}{2}\cdot C_0+\frac{\pi^2}{2}\right)\cdot C_0+\frac{\pi}{2}\cdot T\right]\right\}\equiv C^2,$$

that is, all the possible almost everywhere solutions $u(t,x)$ of problem (1)–(3) are a priori bounded in $B_{2,T}^3$. Theorem is now proved.

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