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GEODESICS ON A PSEUDOSPHERE: ANALYTICAL AND NUMERICAL APPROACHES

Maxsatulloyeva Feruza Maxmud qizi

[1] Faculty of Mathematics, National University of Uzbekistan named after Mirzo Ulugbek

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Abstract

Geodesics on a pseudosphere are examined through analytical and numerical approaches. The pseudosphere, a surface of revolution with constant negative curvature, is characterized using the first fundamental form. The geodesic equations are derived via Christoffel symbols and transformed into a first-order Bernoulli equation. Solutions are obtained both analytically and using Python's LSODA method for numerical integration. Results illustrate the behavior of geodesic trajectories and their dependence on initial conditions. The study provides insights into differential geometry and mathematical physics, highlighting the significance of geodesics in negatively curved spaces.

Keywords: *geodesics, pseudosphere, differential geometry, christoffel symbols, numerical integration, Bernoulli equation*

Introduction

Geodesics, the curves that represent the shortest paths between points on a surface, play a fundamental role in differential geometry and have significant applications in physics, particularly in general relativity and the study of curved spaces.

This work focuses on the analytical and numerical study of geodesics on a pseudosphere. To describe the surface mathematically, we begin with its parametrization and derive the first fundamental form, which characterizes the intrinsic geometry of the surface. Using this foundation, we compute the Christoffel symbols, which are essential in formulating the geodesic equations. These equations, derived from the calculus of variations, describe the natural motion of a particle constrained to move along the surface without external forces. Since the geodesic equations are typically nonlinear second-order differential equations, solving them directly can be challenging. Therefore, we employ a transformation that reduces the geodesic equation into a first-order Bernoulli equation, allowing for more accessible analytical and numerical solutions.

To further analyze the geodesic behavior, we apply numerical methods, specifically the LSODA (Livermore Solver for Ordinary Differential Equations) algorithm, to integrate the equations and visualize geodesic trajectories. LSODA is particularly useful as it adapts between stiff and non-stiff solving methods, ensuring accurate results across different parameter conditions. The numerical solutions provide a detailed exploration of how geodesic paths evolve on the pseudosphere and how their trajectories depend on initial conditions. The results illustrate the fundamental characteristics of motion on negatively curved surfaces, including the divergence of geodesic paths, a hallmark of hyperbolic geometry.

We consider the pseudosphere, which is a surface of revolution formed by rotating a tractrix around its asymptote. Therefore, the pseudosphere, in relation to the tractrix, is given by the following parametrization:

$$x(u,v)=$$

$$=\left(a\cos u\cos v, a\sin u\sin v, a\left(cosu+lntg\frac{u}{2}\right)\right)$$

Then

$$x_u=\left(a\cos u\cos v, a\cos u\sin v, a\frac{\cos^2 u}{sinu}\right)$$

$$x_v=\left(-a\sin u\sin v, a\sin u\cos v, 0\right)$$

For a surface of revolution, the coefficients of the first fundamental form are as follows:

$$E=\frac{a^2\cos^2 u}{\sin^2 u},\ F=0,\ G=a^2\sin^2 u$$

The Christoffel Symbols are:

$$G_{11}^1=-\frac{2}{\sin 2u},\quad G_{12}^1=0,\quad G_{22}^1=-\frac{\sin^3 u}{\cos u},$$

$$G_{11}^2=0,\quad G_{12}^2=ctgu,\quad G_{22}^2=0.$$

The differential equation of geodesic lines on a surface is given as follows:

$$\frac{d^2v}{du^2}=G_{22}^1\left(\frac{dv}{du}\right)^3+\left(2G_{12}^1-G_{12}^2\right)\left(\frac{dv}{du}\right)^2+$$

$$+\left(G_{11}^1-2G_{12}^2\right)\frac{dv}{du}-G_{11}^2$$

Then,

$$\frac{d^2v}{du^2}=-\frac{sin^3u}{cosu}\left(\frac{dv}{du}\right)^3-$$

$$-ctgu\left(\frac{dv}{du}\right)^2-\frac{1+2cos^2u}{cosusinu}\left(\frac{dv}{du}\right)$$

We solve this differential equation for geodesic lines on a pseudosphere. It is a second-order nonlinear differential equation. First, we simplify by letting

$$y=dv/du.$$

This reduces the order of the equation, transforming it into a first-order equation for y. Then,

$$d^2v/du^2=dy/du.$$

The equation becomes:

$$\frac{dy}{du}=-\frac{sin^3u}{cosu}(y)^3-ctgu(y)^2-\frac{1+2cos^2u}{cosusinu}(y)$$

This is a Bernoulli differential equation in y. Let's try the substitution y = 1/z to transform it into a linear equation. After substitution and simplification:

$$z\cdot\frac{dz}{du}=\frac{\sin^3 u}{\cos u}+ctgu\cdot z+\frac{1+2\cos^2 u}{\cos u\sin u}\cdot z^2$$

Bernoulli's equation should be in the following form:

$$\frac{dz}{du}+P(u)z=Q(u)z^n$$

In this case, n = 2, so we write the equation in the following form:

$$\frac{dz}{du}=\frac{\sin^3 u}{z\cos u}+\frac{1}{\tan u}+\frac{\left(1+2\cos^2 u\right)}{\cos u\sin u}z$$

To solve the equation, we use the method of variable substitution. We introduce the following variable: $w=z^{1-n}=z^{-1}$.

Then,

$$\frac{dw}{du}=-z^{-2}\frac{dz}{du}=-\frac{1}{z^2}\frac{dz}{du}$$

We write the equation in terms of w:

$$\frac{dw}{du}-\frac{1}{\tan u}w=-\frac{\sin^3 u}{\cos u}w^2+\frac{\left(1+2\cos^2 u\right)}{\cos u\sin u}$$

Let's solve this equation. Here,

$$P(u)=-\frac{1}{\tan u}$$

$$Q(u)=-\frac{\sin^3 u}{\cos u}w^2-\frac{\left(1+2\cos^2 u\right)}{\cos u\sin u}$$

This is still not a linear equation because there is a w^2 term. If the value of w is very small ($w\approx 0$), the equation becomes linear if we ignore the quadratic term:

$$\frac{dw}{du}-\frac{1}{\tan u}w\approx-\frac{\left(1+2\cos^2 u\right)}{\cos u\sin u}$$

This is a simple linear equation, which we can solve using the following integrals.

Integrating factor method.

General solution of linear equation:

$$I(u)=e^{\int P(u)du}$$

$$P(u)=-\frac{1}{\tan u}, \text{ that is way:}$$

$$I(u)=e^{\int-\frac{1}{\tan u}du}=e^{-\ln|\sin u|}=\frac{1}{|\sin u|}$$

This is used as an integrating factor. Multiplying both sides by $I(u)$:

$$\frac{1}{|\sin u|}\frac{dw}{du}+\frac{1}{\tan u|\sin u|}w=-\frac{\left(1+2\cos^2 u\right)}{\cos u\sin^2 u}$$

Now the left-hand side looks like a complete derivative:

$$\frac{d}{du}\left(w\cdot\frac{1}{\sin u}\right)=-\frac{\left(1+2\cos^2 u\right)}{\cos u\sin^2 u}$$

$$w\cdot\frac{1}{|\sin u|}=\int-\frac{\left(1+2\cos^2 u\right)}{\cos u\sin^2 u}du \ \Rightarrow$$

$$w=|\sin u|\left(\frac{\ln(\sin u+1)}{2}-\frac{\ln(1-\sin u)}{2}-\frac{3}{\sin u}+C_1\right)$$

We had introduced the notation $w = z - 1$, and since $y = z - 1$, it follows that $w = y$. Since $y = dv/du$, the equality becomes:

$$\Rightarrow\frac{dv}{du}=$$

$$|\sin u|\left(\frac{\ln(\sin u+1)}{2}-\frac{\ln(1-\sin u)}{2}-\frac{3}{\sin u}+C_1\right)$$

To find v, we integrate both sides of the equation

$$v=$$

$$=\int|\sin u|\left(\frac{\ln(\sin u+1)}{2}-\frac{\ln(1-\sin u)}{2}-\frac{3}{\sin u}+C_1\right)du$$

Thus, the final result of our integral is as follows:

$$v=-\frac{\cos(u)|\sin(u)|\ln(\sin(u)+1)+\left(-\cos(u)\ln(1-\sin(u))+2C_1\cos(u)+4u\right)|\sin(u)|}{2\sin(u)}+C_2$$

Numerical analysis of Geodesics on a Pseudospherical Surface

The differential equation governing the geodesics is given by:

$$\frac{dy}{du}=-\frac{sin^3u}{cosu}(y)^3-ctgu(y)^2-\frac{1+2cos^2u}{cosusinu}(y)$$

where u represents the parameter along the geodesic, and y denotes the derivative of the geodesic function. This equation describes the behavior of geodesic curves on a pseudospherical surface and is solved numerically using the LSODA method.

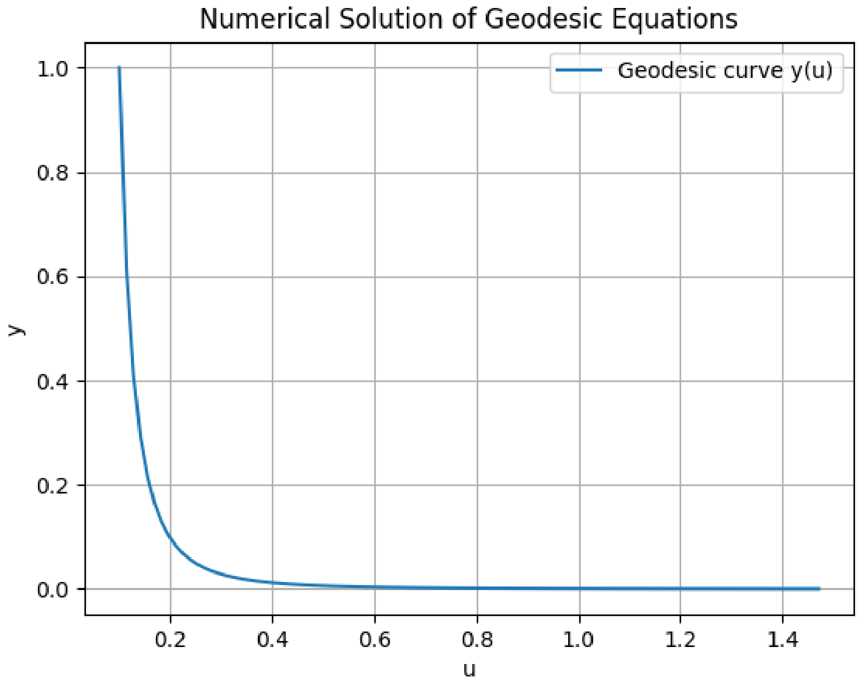
The following Python implementation provides a computational approach to solving and plotting these geodesic curves:

- import numpy as np
- import matplotlib.pyplot as plt
- from scipy.integrate import solve_ivp
- def geodesic_ode(u, y):
- Geodesic equation:
- dy/du = – (sin^3(u)/cos(u)) * y^3 – (cot(u)) * y^2 – ((1 + 2*cos^2(u)) / (cos(u)*sin(u))) * y
- if np.isclose(np.cos(u), 0) or np.isclose(np.sin(u), 0):
- return [0] # Prevent division by zero
- dydu = (–np.sin(u) ** 3 / np.cos(u)) * (y ** 3) \

– (1 / np.tan(u)) * (y ** 2) \
– ((1 + 2 * np.cos(u) ** 2) / (np.cos(u) * np.sin(u))) * y

- return [dydu]
- # Initial conditions
- u0 = 0.1 # Initial parameter value
- y0 = 1 # Initial condition y(0) = 1
- # Calculation interval (avoiding singularities)
- u_span = (u0, np.pi / 2–0.1)
- u_vals = np.linspace(*u_span, 100)
- # Numerical solution using LSODA
- sol = solve_ivp(geodesic_ode, u_span, [y0], t_eval=u_vals,
- method='LSODA')
- # Diagnostic output
- print("Solver success:", sol.success)
- print("Solution shape:", sol.y.shape)
- print("Computed points:", sol.t)
- # Visualization of the geodesic curve

- if sol.success and sol.y.shape[0] > 0:
- plt.plot(sol.t, sol.y[0], label=' Geodesic curve y(u)')
- plt.xlabel('u')
- plt.ylabel('y')
- plt.legend()
- plt.grid()
- plt.title(' Numerical Solution of Geodesic Equations')
- plt.show()



The graph presented here represents the numerical solution of the geodesic equations.

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Contact: feruzamakhsatulloeva@gmail.com