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STUDY OF PENDULUM MOTION USING DIFFERENTIAL EQUATIONS AND THE MAPLE PACKAGE

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Abstract

In this paper, the differential equations of mathematical and physical pendulum motion are solved and investigeted using the Maple application package.

Keywords: *differential equation, mathematical pendulum, physical pendulum, general solution, angular acceleration, mass, moment of inertia*

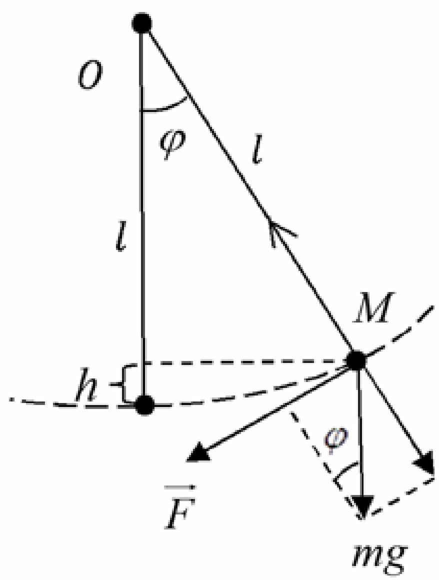
Introduction

The study of mechanical motions leads to the solution of differential equations. Solutions to differential equations cannot always be expressed in terms of elementary functions. Practical packages can be used effectively to test solutions. In the future, we will formulate the equation of motion of the pendulum and explore it using the Maple package. Task statement and solution.

Pendulum motion is used both from a theoretical and practical point of view to solve technological and scientific problems.

A mathematical pendulum is a system consisting of a weightless, inextensible and rod l, suspended at point O, and a material point attached to it with mass m. This pendulum moves in a vertical plane passing through point O. The position of the pendulum is determined by the angle φ between the vertical axis directed downward and the rod, where $\varphi = \varphi(t)$ (t – time).

Figure 1.



According to Newton's second law, the force $\vec{F} = m\vec{a}$ ($\vec{a}$ – is the acceleration vector, $\vec{F}$ is the sum of the tension and gravity forces) acts on the point M with mass m. We project this vector equation onto the tangent to the trajectory at point M. The projection of the tension force is zero and the projection of gravity is $F = mg\sin\varphi$ (see Figure 1). The projection of the acceleration vector is $a = l\varphi''$ (where φ'' is the angular acceleration). Thus,

$$ml\varphi'' = -mg\sin\varphi. \quad (1)$$

This equation is an autonomous second-order differential equation with respect to the unknown function $\varphi = \varphi(t)$. Its order can be lowered by replacing the variable t with φ, which is possible using equation (1):

$$\varphi'' = \frac{d\varphi'}{dt} = \frac{d\varphi'}{d\varphi}\cdot\frac{d\varphi}{dt} = \frac{d}{d\varphi}\left(\frac{\varphi'^2}{2}\right),$$

$$\frac{d}{d\varphi}\left(\frac{\varphi'^2}{2}\right) = -\frac{g}{l}\sin\varphi \text{ or} \quad (2)$$

$$d\left(\frac{\varphi'^2}{2}\right) = -\frac{g}{l}\sin\varphi\, d\varphi$$

Integrating equation (2), we o thain:

$$\frac{\varphi'^2}{2} = \frac{g}{l}\cos\varphi + C_1 \ (C_1 = const).$$

Set in this equation $\omega^2 = \frac{g}{l}$, $C_1 = -\frac{g}{l} + C \quad (C = const)$ and get:

$$\frac{\varphi'^2}{2} + \omega^2(1-\cos\varphi) = C. \quad (3)$$

Equation (3) expresses the conservation of the total energy of the system with mass m, since the kinetic energy is

$$\frac{mv^2}{2} = \frac{m(l\varphi')^2}{2} = ml^2\cdot\frac{\varphi'^2}{2},$$

and potential energy

$$mgh = mgl(1-\cos\varphi).$$

Thus, the total energy of the system:

$$E = \frac{mv^2}{2} + mgh = ml^2\cdot\frac{\varphi'^2}{2} + mgl(1-\cos\varphi) = = ml^2\left(\frac{\varphi'^2}{2} + \frac{g}{l}\cdot(1-\cos\varphi)\right).$$

Therefore, while driving:

$$\frac{\varphi'^2}{2} + u(\varphi) = C, u(\varphi) = \omega^2(1-\cos\varphi)\ (C = const).$$

From here:

$$\varphi' = \pm\sqrt{2(C - u(\varphi))}. \quad (4)$$

In the phase space (φ, φ') this equation determines the phase trajectories.

If $C < 0$, then the trajectory does not exist. If $C = 0$, then $\varphi = \pi k, k \in \mathbb{Z}$ are the equilibrium points of the system.

If $0 < C < 2\omega^2$, then the trajectories will be closed curves corresponding to periodic oscillations of the system.

If $C > 2\omega^2$, then the phase trajectories will be open curves, which corresponds to a rotational motion about the point O.

In general, the solutions of equation (4) cannot be expressed by elementary functions. Solutions can be found using Jacobi elliptic functions.

If in the solution the angle φ varies over a small interval, then $\sin\varphi \approx \varphi$, and (1) the equation will take the form of a harmonic oscillator:

$$\varphi'' + \omega^2\varphi = 0.$$

Solutions of this equation are expressed in terms of elementary functions and have the form:

$$\varphi = A\cos(\omega t + \varphi_0),$$

where A – is the amplitude, φ_0 – is the initial phase. This solution describes harmonic oscillations with period $T = \frac{2\pi}{\omega}$.

Therefore, small oscillations of a mathematical pendulum represent harmonic oscillations.

Let's study the trajectories of the mathematical pendulum in phase space at $\omega = 1$ using the Maple package.

$\omega := 1$

$$\omega := 1$$

$$with(plots) : p1 := implicitplot\left(\frac{(\phi')^2}{2} + \omega\cdot(1-\cos(\phi)) = 1, \phi=-4..4, \phi'=-2..2, scaling = constrained, thickness = 2, color = \text{"Niagara DarkOrchid"}\right)$$

$$p1 := PLOT(...)$$

$$p2 := implicitplot\left(\frac{(\phi')^2}{2} + \omega\cdot(1-\cos(\phi)) = 0.4, \phi=-4..4, \phi'=-2.7..2.7, scaling = constrained, thickness = 2, color = blue\right)$$

$$p2 := PLOT(...)$$

$$p3 := implicitplot\left(\frac{(\phi')^2}{2} + \omega\cdot(1-\cos(\phi)) = 1.98, \phi=-4..4, \phi'=-2.5..2.5, scaling = constrained, thickness = 2, color = green\right)$$

$$p3 := PLOT(...)$$

$$p4 := implicitplot\left(\frac{(\phi')^2}{2} + \omega\cdot(1-\cos(\phi)) = 1.73, \phi=-4..4, \phi'=-2.5..2.5, scaling = constrained, thickness = 2, color = \text{"Generic Blue Purple"}\right)$$

$$p4 := PLOT(...)$$

$$p5 := implicitplot\left(\frac{(\phi')^2}{2} + \omega\cdot(1-\cos(\phi)) = 3.2, \phi=-4..4, \phi'=-2.7..2.7, scaling = constrained, thickness = 2, color = black\right)$$

$$p5 := PLOT(...)$$

$$p6 := implicitplot\left(\frac{(\phi')^2}{2} + \omega\cdot(1-\cos(\phi)) = 2.5, \phi=-4..4, \phi'=-2.5..2.5, scaling = constrained, thickness = 2, color = black\right)$$

$$p6 := PLOT(...)$$

$$p7 := implicitplot\left(\frac{(\phi')^2}{2} + \omega\cdot(1-\cos(\phi)) = 4.1, \phi=-4..4, \phi'=-3.2..3.2, scaling = constrained, thickness = 2, color = black\right)$$

$$p7 := PLOT(...)$$

$display(p1, p2, p3, p4, p5, p6, p7)$

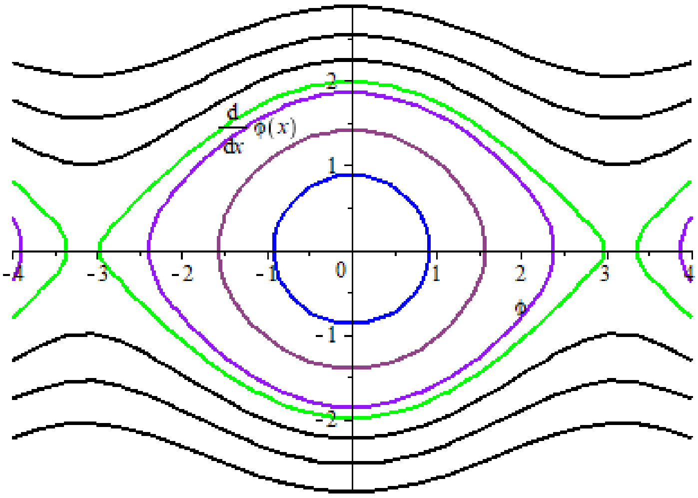


Figure 2.

Let's solve the last equation

$$\varphi' = \pm\sqrt{2\left(c_1 - u(\varphi)\right)}$$

with initial condition $\varphi(0)=\varphi_0, \varphi'(0)=0$.

The solution is expressed in terms of eliptic functions.

We are investigating this solution with Maple. The solution is expressed in terms of the integral:

$$\pm\int_{\varphi_0}^{\varphi} \frac{da}{\sqrt{2\left(u(\varphi_0)-u(a)\right)}} = t.$$

In this case, + is used when the φ' is positive. When pendulum motion starts at an angle φ_0 without an initial velocity, we find its period T.

T period is formed from the following equation

$$-\int_{\varphi_0}^{0} \frac{da}{\sqrt{2\left(u(\varphi_0)-u(a)\right)}} = \frac{T}{4} \quad \text{or}$$

$$\int_{0}^{\varphi_0} \frac{da}{\sqrt{2\left(u(\varphi_0)-u(a)\right)}} = \frac{T}{4}.$$

We compute this period in the mathematical package Maple when $\varphi_0 = \frac{\pi}{3}$.

$$int\left(\frac{4}{\sqrt{2\cdot\left(\cos(\phi)-\cos\left(\frac{\pi}{3}\right)\right)}}, \phi = 0..\frac{\pi}{3}, numeric\right)$$

$$6.743001419$$

Therefore, in a mathematical pendulum, the period of oscillatory motion that began at angle $\frac{\pi}{3}$ under its own weight is

$$T = 6{,}743001419.$$

For a mathematical pendulum in the equation with small deviation angles, the period of oscillation is: T = 27:

$$T = \frac{2\pi}{\omega}.$$

If $\omega = 1$, then the period will be 2π. Obviously, its period is greater than 2π in large movements.

In the same way, we can study the equation of motion of a physical pendulum. A physical pendulum is a solid that oscillates freely under the influence of gravity around an arbitrary horizontal fixed axis that does not pass through the center of gravity.

The equation of motion of a physical pendulum with mass m, rotated from the equilibrium position by an angle φ, is reduced to a differential equation of the form

$\frac{d^2\varphi}{dt^2} + \omega^2 \sin\varphi = 0$, where $\omega^2 = \frac{mgl}{I}$, I is the moment of inertia, l is the distance from the center of gravity of the body to the axis of rotation, g is the acceleration of gravity, m is the mass of the body. Solutions to this differential equation have been studied above.

Conclusion

The equation of small movements of a mathematical pendulum is similar to the equation of harmonic oscillations. Using this equation, you can determine the period of oscillation, angular frequency and other dynamic properties of the pendulum. It follows that the period of small oscillations does not depend on the initial values. However, in the general case of (large oscillations), the oscillation period depends on the initial data.

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