The Austrian Journal of Technical and Natural Sciences, No 1–2
ISSN 2310-5607

PREMIER Publishing
ppublishing.org

Section 3. Mathematics

DOI:10.29013/AJT-25-1.2-45-48

ON A PROBLEM FOR ONE CLASS OF FOURTH ORDER NONLINEAR SOBOLEV TYPE DIFFERENTIAL EQUATIONS

***Samed Aliyev* [1], *Arzu Aliyeva* [2], *Sardar Aliyev* [3]**

[1] Faculty of Mechanics and Mathematics, Department of Methods of Mathematics and its Teaching, Baku State University, Baku, Azerbaijan

[2] Institute of Mathematics and Mechanics Azerbaijcan National Academy of Sciences, Baku, Azerbaijan

[3] Faculty of Mechanics and Mathematics, Department of Higher Mathematics, Baku State University, Baku, Azerbaijan

Cite: *Aliyev S., Aliyeva A., Aliyev S. (2025). On a Problem For one Class of Fourth Order Nonlinear Sobolev Type Differential Equations. Austrian Journal of Technical and Natural Sciences 2025, No. 1–2.* https://doi.org/10.29013/AJT-25-1.2-45-48

Abstract

This work is dedicated to the study of certain properties of the classical solution to a one-dimensional mixed problem for one class of fourth order nonlinear equations. By multiplying the equation under consideration by a suitable function and performing subsequent term by term integration, a theorem on the a priori estimation of the classical solution to the mixed problem is proven.

Keywords: *nonlinear equation, mixed problem, classical solution, a priori estimate*

The study examines certain properties of the classical solution to the following one-dimensional mixed problem:

$$\begin{cases} u_{txx}(t,x) - \alpha\, u_{xxxx}(t,x) = F(t,x,u(t,x),u_x(t,x),u_{xx}(t,x),u_{xxx}(t,x)) \\ \qquad\qquad\qquad\qquad\qquad\qquad (0 \le t \le T,\ 0 \le x \le \pi), & (1) \\ u(0,x) = \phi(x)\ \ (0 \le x \le \pi), & (2) \\ u(t,0) = u(t,\pi) = u_{xx}(t,0) = u_{xx}(t,\pi) = 0 \quad (0 \le t \le T), & (3) \end{cases}$$

where $\alpha > 0$ is a fixed number, $0 < T < +\infty; F$ and ϕ are given functions, and $u(t,x)$ is a sought function.

We call a functio ns $u(t,x)$ a classical solution of the problem (1)–(3) if this function and all its derivatives involved in the equation (1) are continuous in $[0,T]\times[0,\pi]$ and the conditions (1)–(3) are satisfied in the usual sense.

In (Khudaverdiyev K., Aliyeva A., 2010), K. Khudaverdiyev and A. Aliyeva studied the existence of a classical solution to a one-dimensional mixed problem for a certain semilinear equation, which is simpler than equation (1). In (Aliyev S., Heydarova M., Aliyeva A. 2024), proven the existence in small of classical solution of the considered mixed problem. But in (Aliyeva A., 2012), the generalized solution and in (Aliyeva A., 2009), the almost everywhere solution of the considered mixed problem was investigated.

We also note works (Aliyev S., Aliyeva A., Abdullayeva G. 2019; Aliyev S, Aliyeva A., 2017; Tekin I., 2021; Tekin I., 2019), of which some approaches are used in this work.

By multiplying the given equation by an appropriate function and subsequently performing term-by-term integration, the following theorem on the a priori boundedness (in certain senses) of the classical solution to the considered mixed problem is proven.

Theorem 3. Let the right side of equation (1) be as follow:

$$F(t,x,u,u_x,u_{xx},u_{xxx})= \\ = f_0(t,u_{xx})\cdot u_{xxx} + f(t,x,u,u_x,u_{xx},u_{xxx}), \tag{4}$$

where

a) $f_0(t,V)\in C\left([0,T]\times(-\infty,\infty)\right);$ (5)

б) $f(t,x,u_1,...,u_4)\in C([0,T]\times[0,\pi]\times(-\infty,\infty)^4)$ and in $[0,T]\times[0,\pi]\times(-\infty,\infty)^4$

$$f(t,x,u_1,...,u_4)\cdot u_3 \le C\cdot(1+u_1^2+u_2^2+u_3^2)+\delta\cdot u_4^2, \quad 0<\delta<\alpha, \tag{6}$$

where C > 0 is a constant and $\alpha > 0$ is a number appearing in the equation (1).

Then the following a priori estimates hold for all the possible classical solutions $u(t,x)$ of problem (1)–(3):

$$\int_0^\pi u_{xx}^2(t,x)dx \le C_0 \quad \forall t\in[0,T]; \\ \int_0^T\int_0^\pi u_{xxx}^2(t,x)dxdt \le C_0 \tag{7}$$

Proof. Let $u(t,x)$ be any classical solution of problem (1)-(3). On multiplying both sides of equation (1) by the functio $2u_{xx}(t,x)$ and integrating the obtained equality in x over $(0,\pi)$, we get:

$$2\int_0^\pi u_{txx}(t,x)\cdot u_{xx}(t,x)dx - 2\alpha\cdot\int_0^\pi u_{xxxx}(t,x)\cdot u_{xx}(t,x)dx = \\ = 2\int_0^\pi f_0(t,u_{xx}(t,x))\cdot u_{xxx}(t,x)\cdot u_{xx}(t,x))dx + \\ +2\int_0^\pi f(t,x,u(t,x),u_x(t,x),u_{xx}(t,x),u_{xxx}(t,x))\cdot u_{xx}(t,x)dx. \tag{8}$$

Next, using the last two boundary conditions (3) and condition (6), we obtain that $\forall t\in[0,T]$:

$$2\int_0^\pi u_{txx}(t,x)\cdot u_{xx}(t,x)dx = \frac{d}{dt}\left\{\int_0^\pi u_{xx}^2(t,x)dx\right\}; \tag{9}$$

$$-2\alpha\int_0^\pi u_{xxxx}(t,x)\cdot u_{xx}(t,x)dx = \\ = -2\alpha\cdot\left\{\left[u_{xxx}(t,x)\cdot u_{xx}(t,x)\right]_{x=0}^{x=\pi} - \right. \\ \left. -\int_0^\pi u_{xxx}(t,x)\cdot u_{xxx}(t,x)dx\right\} = 2\alpha\cdot\int_0^\pi u_{xxx}^2(t,x)dx; \tag{10}$$

$$2\int_0^\pi f_0(t,u_{xx}(t,x))\cdot u_{xxx}(t,x)\cdot u_{xx}(t,x)dx = \\ = 2\int_0^\pi f_0(t,u(t,x))u_{xx}(t,x)\cdot u_{xxx}(t,x)dx = \\ = 2\int_0^\pi\left\{\frac{\partial}{\partial x}\int_0^{u_{xx}(t,x)} f_0(t,\xi)\cdot\xi\, d\xi\right\}dx = \\ = 2\left\{\int_0^{u_{xx}(t,x)}\xi\, f_0(t,\xi)d\xi\right\}_{x=0}^{x=\pi} = 0; \tag{11}$$

$$2\int_0^\pi f(t,x,u(t,x),u_x(t,x),u_{xx}(t,x),u_{xxx}(t,x))\cdot u_{xx}(t,x)dx \le \\ \le 2\int_0^\pi\left\{C\cdot[1+u^2(t,x)+u_x^2(t,x)+u_{xx}^2(t,x)]+\delta\cdot u_{xxx}^2(t,x)\right\}dx = \\ = 2\pi\cdot C + 2C\cdot\int_0^\pi u^2(t,x)dx + 2C\cdot\int_0^\pi u_x^2(t,x)dx + \\ +2C\cdot\int_0^\pi u_{xx}^2(t,x)dx + 2\delta\cdot\int_0^\pi u_{xxx}^2(t,x)dx. \tag{12}$$

Now, substituting (9)–(12) into (8), integrating the resulting inequality from 0 to t, and using the initial condition (2), we obtain that $\forall t\in[0,T]$:

$$\int_0^\pi u_{xx}^2(t,x)dx + 2\alpha \cdot \int_0^t\int_0^\pi u_{xxx}^2(\tau,x)dxd\tau \le$$
$$\le \int_0^\pi (\phi''(x))^2 dx + 2\pi \cdot C \cdot T +$$
$$+\int_0^t \left\{ 2C \cdot \left[\int_0^\pi u^2(\tau,x)dx + \int_0^\pi u_x^2(\tau,x)dx + \int_0^\pi u_{xx}^2(\tau,x)dx \right] + \right.$$
$$\left. +2\delta \cdot \int_0^\pi u_{xxx}^2(\tau,x)dxd\tau \right\},$$

consequently,

$$\int_0^\pi u_{xx}^2(t,x)dx + 2(\alpha-\delta) \cdot \int_0^t\int_0^\pi u_{xxx}^2(\tau,x)dxd\tau \le$$
$$\le \int_0^\pi (\phi''(x))^2 dx + 2\pi T \cdot C + \tag{13}$$
$$+2C \cdot \int_0^t \left\{ \int_0^\pi u^2(\tau,x)dx + \int_0^\pi u_x^2(\tau,x)dx + \int_0^\pi u_{xx}^2(\tau,x)dx \right\} d\tau.$$

Next, since $u(\tau,0) = u(\tau,\pi)$ $(0 \le \tau \le T)$, then $\forall \tau \in [0,T]$ there exists such a point $\xi = \xi_\tau \in (0,\pi)$, that $u_x(\tau,\xi_\tau) = 0$. Then it is obvious that $\forall \tau \in [0,T]$ and $x \in [0,\pi]$:

$$u_x(\tau,x) = \int_{\xi_\tau}^x u_{\xi\xi}(\tau,\xi)d\xi ,$$
$$u_x^2(\tau,x) \le \left\{ \int_0^\pi \left| u_{\xi\xi}(\tau,\xi) \right| d\xi \right\}^2 \le \tag{14}$$
$$\le \pi \cdot \int_0^\pi u_{\xi\xi}^2(\tau,\xi)d\xi = \pi \cdot \int_0^\pi u_{xx}^2(\tau,x)dx$$

$$\int_0^\pi u_x^2(\tau,x)dx \le \pi \cdot \int_0^\pi u_{xx}^2(\tau,x)dx \cdot \pi = \tag{15}$$
$$= \pi^2 \cdot \int_0^\pi u_{xx}^2(\tau,x)dx .$$

On the other hand, using the relation $u(\tau,0) = 0$ $(0 \le \tau \le T)$, we obtain $\forall \tau \in [0,T]$ and $x \in [0,\pi]$:

$$u(\tau,x) = \int_0^x u_\xi(\tau,\xi)d\xi ,$$
$$u^2(\tau,x) \le \left\{ \int_0^\pi \left| u_\xi(\tau,\xi) \right| d\xi \right\}^2 \le \tag{16}$$
$$\le \pi \cdot \int_0^\pi u_\xi^2(\tau,\xi)d\xi = \pi \cdot \int_0^\pi u_x^2(\tau,x)dx ,$$

$$\int_0^\pi u^2(\tau,x)dx \le \pi \int_0^\pi u_x^2(\tau,x)dx \cdot \pi = \tag{17}$$
$$= \pi^2 \cdot \int_0^\pi u_x^2(\tau,x)dx .$$

Then, using a priori estimate (15), we obtain from (16) and (17), that $\forall \tau \in [0,T]$ and $x \in [0,\pi]$:

$$u^2(\tau,x) \le \pi \cdot \int_0^\pi u_x^2(\tau,x)dx \le \pi^3 \cdot \int_0^\pi u_{xx}^2(\tau,x)dx , \tag{18}$$

$$\int_0^\pi u^2(\tau,x)dx \le \pi^3 \cdot \int_0^\pi u_{xx}^2(\tau,x)dx \cdot \pi = \tag{19}$$
$$= \pi^4 \cdot \int_0^\pi u_{xx}^2(\tau,x)dx .$$

Now, using notation

$$\delta_0 \equiv 2(\alpha-\delta) > 0$$

and estimate (19), (15) on the right-hand side of (13), from (13) we obtain that $\forall t \in [0,T]$:

$$\int_0^\pi u_{xx}^2(t,x)dx + \delta_0 \cdot \int_0^t\int_0^\pi u_{xxx}^2(\tau,x)dxd\tau \le$$
$$\int_0^\pi (\phi''(x))^2 dx + 2\pi T \cdot C + \tag{20}$$
$$+2C \cdot (\pi^4 + \pi^2 + 1) \cdot \int_0^t \left\{ \int_0^\pi u_{xx}^2(\tau,x)dx \right\} d\tau .$$

Then, due to $\delta_0 > 0$, it is obvious that $\forall t \in [0,T]$:

$$\int_0^\pi u_{xx}^2(t,x)dx \le$$
$$\int_0^\pi (\phi''(x))^2 dx + 2\pi T \cdot C + 2(1+\pi^2+\pi^4) \cdot C \times$$
$$\times \int_0^t \left\{ \int_0^\pi u_{xx}^2(\tau,x)dx \right\} d\tau .$$

Hence, applying Bellman's inequality we obtain that $\forall t \in [0,T]$:

$$\int_0^\pi u_{xx}^2(t,x)dx \le \left\{ \int_0^\pi (\phi''(x))^2 dx + 2\pi T \cdot C \right\} \times \tag{21}$$
$$\times \exp\left\{ 2(1+\pi^2+\pi^4) \cdot C \cdot T \right\} \equiv C_1 .$$

Then, using a priori estimate (21), we obtain from (20) that $\forall t \in [0,T]$:

$$\delta_0 \cdot \int_0^t\int_0^\pi u_{xxx}^2(\tau,x)dxd\tau \le$$
$$\int_0^\pi (\phi''(x))^2 dx + 2\pi T \cdot C + 2(1+\pi^2+\pi^4) \cdot C \cdot C_1 \cdot T \equiv C_2 .$$

Consequently,

$$\int_0^T\int_0^\pi u_{xxx}^2(\tau,x)dxd\tau \le \delta_0^{-1} \cdot C_2 \equiv C_3 . \tag{22}$$

Now, from (21) and (22) it follows the trueness of a priori estimates (7). Theorem is now proved.

In conclusion, this work serves as a continuation of (Aliyev S., Aliyeva A., 2020), where the a priori boundedness (in a certain sense) of the almost everywhere solution considered mixed problem was studied.

References

Aliyeva A. Investigation of generalized solution of one-dimensional mixed problem for a class of fourth order semi linear equations of Sobolev type, Transactions of National Academy of Sciences of Azerbaijan, – V. XXXII. – No. 4. – Baku, 2012. – P. 3–12.

Aliyeva A. On the existence in large for almost everywhere solution of one-dimensional mixed problem for a class of semilinear fourth order equations of Sobolev type, Proceedings of Institute of Mathematics and Mechanics, – V. XXX. 2009. – P. 19–36.

Aliyev S., Aliyeva A., Abdullayeva G. On the existence of solution to multidimensional third order nonlinear equations, European Journal of Pure and Applied Mathematics – 12 (2). 2019. – P. 577–589.

Aliyev S., Aliyeva A. On the existence for almost everywhere solution of multidimensional mixed problem for one class third order differential equations with nonlinear operator in the right-hand side, International Journal of Pure and Applied Matematics – 115 (3). 2017. – P. 549–560.

Aliyev S., Aliyeva A. The investigation of one-dimensional mixed problem for one class of nonlinear fourth order educations, European Journal of Technical and Natural Sciences, No. 2. 2020. – P. 16–18.

Aliyev S., Heydarova M., Aliyeva A. On the existence of classical solution to one-dimensional fourth order semilinear equations, Advances in Differential Equations and Control Processes – 31 (2). 2024. – P. 165–185.

Khudaverdiyev K., Aliyeva A. On the global existence of solution to one-dimensional fourth order nonlinear Sobolev type equations, Applied Mathematics and Computation, – V. 217. 2010. – P. 347–354.

Tekin I. Inverse problem for a nonlinear third order in time partial differential equation. Mathematical Methods in the Applied Sciences, – 44 (11). 2021. – P. 9571–9581.

Tekin I. Determination of a time-dependent coefficient in a wave equation with unusual boundary condition. Filomat, – 33 (9). 2019. – P. 2653–2665.

submitted 06.01.2025;
accepted for publication 20.01.2025;
published 28.03.2025

Contact: samed59@bk.ru